Systems, Generativity and Interactional Effects

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Abstract

The principal concern of the thesis is to understand and uncover interaction-related effects—termed, *interactional effects*—that arise from the interaction of systems. We may describe the common situation of interest as small entities of systems coming together, interacting, and producing as an aggregate a behavior that would not have occurred without interaction. Those situations are fundamental and appear in countless settings, particularly in settings exhibiting cascade-like intuition. The goal of the research is to show that one can *extract* from a system the potential it has to generate such effects, and use those extracts to reconstruct, or characterize, the phenomena that emerge upon interaction. In technical terms, we expose cascade-like effects as a loss of exactness, and reveal them as (co)homological intuition waiting to be formalized.

We propose a means to relate properties of an interconnected system to its separate component systems in the presence of cascade-like effects. Building on a theory of interconnection reminiscent of the behavioral approach to systems theory, we introduce the notion of *generativity*, and its byproduct, *generative effects*. Cascade effects are seen as instances of generative effects. The latter are precisely the instances where properties of interest are not preserved or behave very badly when systems interact. The work overcomes that obstruction. We show how to extract algebraic objects (e.g., vectors spaces) from the systems, that encode their generativity: their potential to generate new phenomena upon interaction. Those objects may then be used to link the properties of the interconnected system to its separate systems. Such a link will be executed through the use of exact sequences from commutative algebra.

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I cannot genuinely elucidate my experiences throughout the thesis work. Time has eroded much of their vivacity from my recollection. The past years have also been drenched by many tumultuous intertwined happenings. The most simple experiences—and often the essential ones—are thus very likely to be brought very little to light. It is a very unfair nature of matters. As such, I chose to be succinct with little colors and quite minimal. There is then much that is knowingly left untold and holds quite a nice place in my heart. Many individuals will also remain unthanked. My experiences, however, would not have been the same without each of them, in his or her (even little) way.

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Chapter 1

Introduction

The principal concern of the thesis is to understand and uncover interaction-related effects—termed, *interactional effects*—that arise from the interaction of systems. We may describe the common situation of interest as small entities of systems coming together, interacting, and producing as an aggregate a behavior that would not have occured without interaction. Those situations are fundamental and appear in countless settings, two of which are contagion effects in societal systems, and cascading failures in infrastructures. The goal of the research is to show that one can *extract* from a system the potential it has to generate such effects, and use those extracts to reconstruct, or characterize, the phenomena that emerge upon interaction.

The thesis proposes a means to relate properties of an interconnected system to its separate component systems in the presence of cascade-like effects. Building on a theory of interconnection reminiscent of the behavioral approach to system theory, it introduces the notion of *generativity*, and its byproduct, *generative effects*. Cascade effects, enclosing contagion phenomena and cascading failures, are seen as instances of generative effects. The latter are precisely the instances where properties of interest are not preserved or behave very badly when systems interact. The goal of the work is to overcome that obstruction. We show how to extract algebraic objects (e.g. vectors spaces) from the systems, that encode their generativity: their potential to generate new phenomena upon interaction. Those objects may then be used to link the properties of the interconnected system to its separate systems. Such a link will be executed through the use of exact sequences from commutative algebra.

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1.1 The phenomenon and the problem.

The work begins by understanding *cascade effects.* An intuitive instance of such effects is embodied by a trail of falling dominos. The fall of a domino triggers the fall of its successor in the sequence. If the first domino falls, then the whole sequence of dominoes collapses by induction. On a more *pressing* end, these effects pervade exinctions in ecosystems, spreads of epidemics, financial crises, power blackouts, cascading failures of infrastructures, propogation of delays, societal adoption trends and diffusion of innovation. Those effects prove to be detrimental in some instances, and a potential to harness in others. There is quite an interest in understanding such effects, and an everincreasing need to do so. Many (mathematical) models are proposed to handle such situations, however we seem to be rather far from a fundamental (universal) understanding of the phenomenon. For one, most of the intuition floating around is still intangible, and a good number of essential words are detached from a clear natural meaning. The term *cascade effects* is itself informal, vague and prone to many interpretations. Nevertheless, some intuition does exist to give rise to the term. The question is then how to capture that intuition mathematically in a universal way.

So we begin by asking: what are cascade effects? And where do they come from? For the purpose of answering the question, we will momentarily interpret the term cascade effects broadly. We will then replace it with the term of *interactional effects*. Intuitively, yet informally, those are phenomena that emerge from the interaction of several systems. The thesis will be concerned with interactional effects that are *generative* in nature, termed *generative effects*. Those will be argued, later on in the thesis, to enclose the informal cascading phenomena. The thesis will also formally argue that interactional effects may in general be cast as generative effects.

1.1.1 An analogy.

We will anecdotedly illustrate the emergence of interactional effects through the emergence of magic. We are referring to magic that arises, from a performing art, from either parlor tricks or sleight of hand. Although the analogy is anecdotal, the thesis formalizes it later on. In a nutshell, there is no magic for the performing magician, there can only be magic for the observing audience. For convenience, we will have our performing **M**agician go by the name Merlin, and our observing **A**udience go by the name Arthur.

Merlin skillfully performs to Arthur his most intricate magical trick in his repertoire. Arthur is impressed, as he should be, and senses some magic. Merlin now performs the same trick again to Arthur, but he performs it slowly. Arthur begins to see more than what he had seen before. Somehow the magic starts to go away. Merlin may even go further in explaining exactly what he is doing. The magic then dwindles, up to a limit point where it does not exist anymore. However, throughout the performance, the trick itself has not changed at all. The only thing that changed is the ability for Arthur to see initially-hidden things. To further expound, if Merlin performed his trick again, in all its quickness, yet behind an opaque curtain, then Arthur will see nothing. There can be no magic for Arthur in this case. Merlin can also vary the amount of magic exhibited by the trick by concealing some parts of it and making explicit some other parts.

Indeed, the reason magic appears in Merlin's first performance is because Arthur cannot perceive everything. The dexterity of Merlin and the intricate design of the trick cast a veil on the workings of the trick. Dually, Arthur's humane lack of supernatural abilities to see and understand casts a veil on the trick. There are things that Arthur can observe, and there are other things that are hidden to him and cannot be seen. Athur then builds up an expectation based on what he can see. He then declares magic whenever he cannot explain the discrepancy between his expectation and the actual happening.

The veil on the trick is Arthur's inability to observe everything. The veil causes magic to emerge. But there is also an underlying ingredient: that of Merlin performing the trick and Arthur building his expectation. The trick is seen to consist of several small pieces of tricks combined. The big trick is performed by combining or interconnecting the small pieces tricks, causing them to interact. Magic then emerges as follows. Arthur recognizes the small tricks, and creates an expectation of what each ought to perform. On one end, he combines the small tricks devoid of magic, namely his small expectations, and builds a big expectation on the whole trick. On another end, he observes the final development of Merlin's trick. He then compares his big expectation to the outcome of Merlin's trick. If there is a discrepency, there is magic. If there is no discrepency, Arthur is unmoved and is the least confused.

Back to the systems.

How does the anecdote cast back into our setting? The tricks will be our systems, with all their complexities. We then cast a veil on the space of systems. The veil will either conceal mechanisms in a system or hide some of its characteristics. What remains visible is observable, and is termed *the phenome*. Phenomes ought to be thought of as simplified systems. Those are Arthur's expectations. Interactional effects are then said to emerge whenever the phenome of the interconnected system, cannot be explained by interconnecting or combining the phenomes of the separate systems.

1.1.2 The mathematical sketch.

Thus a definition of interactional effects would need (at least) two ingredients. The first is a notion of interaction or interconnection of systems. However, such a notion by itself cannot give rise to interactional effects. The interconnection of two systems only gives an interconnected system, and nothing more. The second ingredient then consists of equipping the theory of interconnection with a notion of interactional-effects. Such effects emerge once we conceal features from a systems, by declaring what is observable from the system. The ingredients may be summarized through the diagram:

$$\begin{tikzcd} \langle P, + \rangle & \langle S, \oplus \rangle \arrow[l, "\Phi"'] \end{tikzcd}$$

The space $\langle S, \oplus \rangle$ represents the space of systems. For simpliciy, S is a set, and $\oplus$ is a binary operator on S. The sum $s_1 \oplus s_2$ denotes the interaction of s_1 and s_2. The map Φ represents the veil, sending a system to its observable part, its phenome, in P. The set P ought to be thought of as a space of simplified systems, and thus gains a notion of interaction through the + operator. Of course, the elements of this diagram are not arbitrary, and the complete mathematical diagram is slightly more intricate. We will expound it in the next subsection.

Regardless, interactional effects are then said to emerge whenever:

$$\Phi(s_1 \oplus s_2) \neq \Phi(s_1) + \Phi(s_2)$$

Interactional effects are sustained by the veil whenever the phenome of the combined system cannot be explained by the phenome of the separate systems. **The inequality we obtain is then fundamental.** If the inequality is not present, our intuition for interactional phenomena vanishes. If the inequality is present, the intuition emerges.

Interactional effects are then encoded in the inequality, or more precisely in the properties of the veil. The fundamental question is then: how do we cope with the inequality? How do we mend the inexactness. We thus view the fundamental question as to how to uncover interactional effects emerging from the interaction of systems. Relatedly, how can we relate the phenome of the interconnected system to that of its separate parts despite the presence of interactional effects? We deem this question fundamental as it does not dissolve in the most general setting we can think of capturing the intuition we're after. Yet, even in this generality we can obtain good answers. To formally answer the question, we would first need to give a summarized formal account of the question, and the work.

1.2 The summary

We define a *sandbox* to be a pair $(\mathcal{S}, \leq)$ of a set $\mathcal{S}$ and a transitive binary relation $\leq$ on $\mathcal{S} \times \mathcal{S}$. Transitivity means that if $a \leq b$ and $b \leq c$, then $a \leq c$. The set $\mathcal{S}$ is interpreted to be a set of systems, and $a \leq b$ indicates that a is a subsystem of b. The relation $\leq$ enables systems to interact in the following manner. We define an interact operation $\vee : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ that sends a pair of systems (a, b) to the *smallest* system containing both a and b as subsystems. The $\vee$ operation embodies that a system resulting from the interaction, or interconnection, of a and b ought to contain both a and b as parts, and is a subsystem of any other system containing both a and b. Such an operation may not be defined, but we assume, for this overview, that the relation $\leq$ is such that the operation is well defined for every pair of systems.

We are then given a sandbox $(\mathcal{S}, \leq)$ and an interact operation $\vee$. Those objects define our theory of interconnection. Yet, our theory of interconnection is not enough to produce interactional effects. The interactional effects can only emerge once we decide to conceal either mechanisms or characteristics of systems.

We define a quasi-veil to be a pair $(\Phi, \mathcal{P})$ of a sandbox $(\mathcal{P}, \sqsubseteq)$ and a map $\Phi : \mathcal{S} \to \mathcal{P}$ such that $a \leq b$ implies $\Phi(a) \sqsubseteq \Phi(b)$, i.e., preserving the subsystem relation. The quasi-veil is intended to partially cover the sandbox, leaving the systems in play partially observed. We interpret $(\mathcal{P}, \sqsubseteq)$ as the sandbox of phenomes, of what in the systems is chosen to be observable to us. By choosing what we wish to observe, we automatically choose what we wish to conceal. As $(\mathcal{P}, \sqsubseteq)$ is a sandbox, it acquires an interact operator $\sqcup$. Interactional effects are said to have emerged, or to be *sustained by the quasi-veil*, during the interaction of a and b, when:

$$\Phi(a \vee b) \neq \Phi(a) \sqcup \Phi(b). \tag{1.1}$$

Interactional effects would have emerged when the phenome of the separate systems, cannot explain the phenome of interconnected system. A great deal of situations may be put into this form.

We then ask the following question, considered in the thesis at almost the same structural level:

Question. *Given a sandbox $(\mathcal{S}, \leq)$ and a quasi-veil $(\Phi, \mathcal{P})$ that sustains interactional effects, how can we express or characterize $\Phi(a \vee b)$ non-trivially through $\Phi(a)$, $\Phi(b)$ and other information on a, b, and potentially a common subsystem $a \wedge b$.*

How can we relate the behavior of the interconnected system to that of its subsystems despite the presence of interactional effects? Although the mathematical answer to the question may appear cryptic if one is not familiar with the terms, we provide it for completeness. We nevertheless illustrate it with a simple accessible example.

The Answer. *We can generally construct a cohomology theory whose zeroth-order terms encode the phenome. The higher-order cohomology objects encode the systems' potential to produce interactional effects, and may be used to relate the phenome of the interconnected system to its separate system.*

As promised, we illustrate the answer through an example.

1.2.1 The example.

Given two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n'}$:

$$\begin{bmatrix} A \end{bmatrix} \qquad \begin{bmatrix} B \end{bmatrix},$$

our goal is to understand the kernel of the $m \times (n + n')$ matrix:

$$\begin{bmatrix} C \end{bmatrix} := \begin{bmatrix} A & B \end{bmatrix}$$

consiting of the concatenation of the separate matrices. The matrix C *inherits* the kernels of both A and B, but A and B interact so as to generate new elements in the kernel of C. In general, we have:

$$\ker C \neq \ker A \oplus \ker B.$$

Of course we can simply compute the kernel of the big matrix, yet we get no insight on the interaction. But our theory give us that:

$$\ker C = \ker A \oplus \ker B \oplus (\operatorname{im} A \cap \operatorname{im} B).^{1}$$

The elements of $\ker A \oplus \ker B$ are already known to be part of the kernel from the separate systems. The newly generated elements are encoded in $\operatorname{im} A \cap \operatorname{im} B$ and correspond to the interaction of the two matrices. The theory then tells us that we further need only to keep the images of the separate matrices to characterize $\ker C$.

[1] Equality here is an isomorphism. The space $(\operatorname{im} A \cap \operatorname{im} B)$ is not a subspace of $R^{n+n'}$, but every element of it corresponds to an equivalence class in $\ker C$ modulo $(\ker A + \ker B)$. The dimensions of the two sides are equal. But we cannot automatically deduce a basis for $\ker C$ from the characterization. We can deduce a basis for $\ker C$ if we have a basis for $\operatorname{im} A \cap \operatorname{im} B$ in R^m and collect, for each element of it, one preimage with respect to A and B separately (any preimage would do).

One obviously cannot reconstruct a matrix only knowing its image and its kernel. In the case where the matrices are very fat, i.e., $m \ll n$, then keeping the image yields a major computational benefit.

1.2.2 As sandboxes and quasi-veils.

We can cast the example through sandboxes and quasi-veils. Let $\mathcal{C}$ be a finite subset of $\mathbb{R}^m$, and define $\mathcal{S}$ to be powerset of $\mathcal{C}$. We then obtain a sandbox $(\mathcal{S}, \subseteq)$, where the interact operation $\cup$ is set union. A system $\{c_1, \cdots, c_k\}$ of $\mathcal{S}$ defines a unique matrix with columns $c_1, \cdots, c_k$, up to column reordering. We will fix a unique order by ordering the elements of $\mathcal{C}$. Given two system A and B, their interaction yields a system, consisting of forming the unique matrix with the columns in $A \cup B$. Thus if A and B are disjoint, then $A \cup B$ yields the concatenation of the separate matrices.

The sandbox of phenomes $(\mathcal{P}, \subseteq)$ is the set of linear subspaces of $\mathcal{R}^{|\mathcal{C}|}$ ordered by inclusion, where the interact operation $+$ is set union followed by the linear span closure. The veil Φ sends a set A to the kernel of its corresponding matrix, which can be seen as an element of $\mathcal{P}$. The map Φ is obviously inclusion preserving. We generally get:

$$\Phi(A \cup B) \neq \Phi(A) + \Phi(B)$$

If A and B are disjoint sets, our theory gives us:

$$\Phi(A \cup B) \neq (\Phi(A) + \Phi(B)) \oplus (\operatorname{im} A \cap \operatorname{im} B)$$

If A and B are not disjoint sets, then we define $D = A \cap B$ and our theory gives us:

$$\Phi(A \cup B) \neq (\Phi(A) + \Phi(B)) \oplus (\operatorname{im} A \cap \operatorname{im} B)/ \operatorname{im} D$$

Our theory defines veils instead quasi-veil. Those are only quasi-veils with an additional property. We do not need to be explicit about this property in this overview. But once veils are defined, we term our interactional effects as generative effects.

1.2.3 Roadmap to solution.

To arrive at such a solution, or more generally to the answer to our question, we need to generalize our sandboxes. We do so by replacing a relation $a \leq b$ by several morpshims, or functions, $a \to b$. By doing so, we arrive at the notion of a category. The interaction operator then generalizes to the notion of colimit, or inductive limit.

When the systems possess a linear (or *abelian*) property, the key property that we use is exactness, which is borrowed from commutative algebra. The steps needed to arrive at an answer can be illustrated through the following correspondences:

Interaction of systems	=	Short exact sequences
Quasi-Veil	=	Functor
Veil	=	Left-Exact Functor
Interactional Effects	=	Loss of exactness on the right
Extra information needed	=	Derived functors (or cohomology objects)
Relating big system to parts	=	Long exact sequence

We can then form a longer exact sequence relating the phenome of the interconnected system to that of its subsystems. Characterizations of the sort provided are then deduced from the exact sequence. Of course, not all systems are abelian. We will thus need to lift our sandbox to a sandbox that contains abelian systems. For instance, cardinality of finite sets may be encoded in the dimenions of the vector spaces.

It is nevertheless possible to avoid an explicit generalization. Such a feat might provide a more accessible presentation, but will come at the expense of keeping the mathematical intuition mysterious and the results somewhat out-of-the-blue.

1.2.4 A non-contrived solution to the matrix example.

We provide a full non-contrived solution to the posed matrix problem for completeness. This section assumes that the reader has some familiarity with the concepts herein. Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$, $T_B : \mathbb{R}^{n'} \to \mathbb{R}^m$ and $T_C : \mathbb{R}^{n+n'} \to \mathbb{R}^m$ denote the linear maps associated to the matrices A, B and C, respectively. We define T_0 to be the linear map $0 \to \mathbb{R}^m$. The *interconnection* of matrices:

$$\begin{bmatrix} C \end{bmatrix} := \begin{bmatrix} A & B \end{bmatrix}$$

consists of taking a *pushout* (or *colimit*) along the diagram:

$$T_A \longleftarrow T_0 \longrightarrow T_B$$

The arrows $T_0 \to T_A$ and $T_0 \to T_B$ correspond to the diagrams:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "T_0"] & \mathbb{R}^n \arrow[d, "T_A"] \\
\mathbb{R}^m \arrow[r, "id"] & \mathbb{R}^m
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] \arrow[d, "T_0"] & \mathbb{R}^{n'} \arrow[d, "T_B"] \\
\mathbb{R}^m \arrow[r, "id"] & \mathbb{R}^m
\end{tikzcd}$$

The pushout then yields a commutative ladder diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d, "T_0"] & \mathbb{R}^n \oplus \mathbb{R}^{n'} \arrow[r] \arrow[d, "{(T_A,T_B)}"] & \mathbb{R}^{n+n'} \arrow[r] \arrow[d, "T_C"] & 0 \\
0 \arrow[r] & \mathbb{R}^m \arrow[r, "{(id,-id)}"] & \mathbb{R}^m \oplus \mathbb{R}^m \arrow[r, "{(a,b)\mapsto a+b}"] & \mathbb{R}^m \arrow[r] & 0
\end{tikzcd}$$

The ladder diagram with exact rows is only an exact sequence of linear maps. We may apply the Snake lemma to the diagram above, and recover a long exact sequence:

$$0 \to 0 \to \ker T_A \oplus \ker T_B \to \ker T_C \to \mathbb{R}^m \to \mathbb{R}^m / \operatorname{im} T_A \oplus \mathbb{R}^m / \operatorname{im} T_B \to \mathbb{R}^m / \operatorname{im} T_C \to 0$$

The characterization can then be easily deduced from the exact sequence. The ker functor is additive and left exact. It admits right derived functors, where only the first is non-trivial and corresponds to the coker functor. We refer to Chapter 7 or 8 (equivalently, [Ada17f] or [Ada17g]) for the details.

The common part T_0 may be modified accordingly to yield different characterizations. Furthermore, different patterns of interconnection can be recreated, and a similar procedure would apply.

1.3 Listing the Contribution.

As described, the notion of *generative effects*, explained by the notion of *generativity*, captures the intuition of cascading effects and contagion phenomena. We use the term *interactional effects* to refer to other potential interaction-related effects that need not emerge from generativity.

1.3.1 Key contribution.

∗∗∗ Generative effects is loss of exactness.

Specifically, cascading and/or contagion phenomena arise from a loss of exactness. This claim requires technical development and intuition to support it. Once supported, it entails a rich direction of theoretical pursuit as a corollary.

1.3.2 Supporting contribution.

∗∗∗ We formalize interconnection of systems through joins in semilattices on a special level, and colimits in categories on a general level. We provide a systems-theoretic interpretation to such interconnection, by revealing that our approach coincides with the abstract structure of the behavioral approach, and revealing connections to typical models used. Our view allows a clear definition of subsystems and controlled systems, and enables a sound separation between syntax and semantics, akin to the separation of behavioral equation representations and behaviors.

∗∗∗ We introduce the notion of generativity, and its byproduct generative effects. Cascade effects, enclosing contagion phenomena and cascading failures, are seen as instances of generative effects. Generative effects emerge from a mix of concealing mechanisms and concealing characteristics in a system. Features are concealed through the use of a veil, that sets up an adjointness between the space of systems and the space of phenomes, namely the space of what is visible in the systems. Generative effects emerge when the functor underlying the veil fails to commute with colimits.

1.3.3 Corollary contribution.

*** We introduce the notion of exact sequences, or more generally exactness in abelian settings, into the setting of systemic interaction and interactional effects:

- as a means to express interconnection, or interaction, of systems.
- as a means of relating the behavior of a system to its separate components.

The idea of exactness is very prevalent in algebraic mathematical disciplines, notably algebraic topology and geometry. The idea can further be extended to non-abelian settings.

*** We introduce generative effects as a functorial phenomenon, meaning one that may only appear when systems are related together, say through morphism. We show that it translates to a loss of exactness. Specifically, in abelian settings, generative effects emerge when the veil fails to send surjective maps to surjective maps.

*** We introduce homological methods and show how cascade-like phenomena, in setting of abelian systems, is really cohomological intuition waiting to be formalized:

- We extract *universal* (cohomology) objects that encode the system's potential to generate effects.
- We use the objects to understand and characterize the behavior generated from the interaction of systems.
- We use the objects to relate the behavior of the interconnected system to that of its subsystems.

Those objects allow us to understand the role played by the concealed mechanisms upon the interaction of systems.

Remark. Homological methods in abelian settings are the most direct way to get such a characterization. There are means to generalize those, for instance through homotopical methods or non-abelian methods. The thesis will only be concerned with homological methods, in their simplest direct form.

*** We introduce the idea of linearizing generative effects. Cascading phenomena do not occur from non-linearities. In fact, they are completely orthogonal to them. We prove that lifting a problem to an abelian setting is always possible. The question then becomes: can we find a *good* lift?

1.3.4 Case contribution.

*** We reveal insight and connections into the following instances:

- Cascading phenomena through the lens of the behavioral approach.
- Contagion phenomena as depicted by networked models.

- Reachability problems, cascading failures and paths synchronization.
- Interconnection and the effects of memory.
- Problems in *interconnecting* matrices.

Each instance will yield different results. As an example, in the case of LTI systems and memory, we uncover the notion of lag of a system as the primary proponent that captures the role of memory that is played during interaction.

*** We introduce a class of systems that captures the notion of deduction-like contagion effects. The analysis of that class evolves as an interplay of lattices and fixed-points. This class provides the archtypical example of generative effects.

*** We provide various abelian lifts for various classes of situations.

1.4 Discussion on the contribution.

The contribution of the thesis may be laid on three fronts: a conceptual contribution, an actionable contribution and a linking contribution.

1.4.1 The conceptual side.

A great deal of effort is currently put by the research community into understanding the behavior of complex systems. The continual advent of financial crises, power blackouts, and socio-political turmoil spillovers has made it even more pressing to understand these cascade-like interactional effects leading to these phenomena. A great deal of models are proposed in the literature to address those issues, however we seem to be rather far from a universal understanding of the essence of these phenomena. For one, most of the intuition floating around is still intangible, and a good number of essential words are detached from a clear natural meaning.

The conceptual side of the thesis aims to fill that gap. The theory developed serves as a meta-theory. It lives outside of mathematical models, and may be instantiated to details as pleased. The theory builds a theory of interconnection, and formalizes the elusive notion of cascading phenomena through generativity. The theory tells us that we can *measure* the potential for cascading phenomena to occur. That potential is extracted—and thus hereafter termed *extract*—from the systems, and can be used to link the behavior of a complicated system to its subsystems despite the presence of the obstructive interactional effects. The extracts are shown to be universal in a certain sense, and tend to be much smaller than the systems. They distill the *cause* of the emerging behavior.

1.4.2 The actionable side.

The goal when dealing with the interaction of systems is *coping* with the phenomena that emerge therein. Such a cope manifests itself on various level. Our ability to tame the phenomena, be it through forecasting or control, relies first on our ability to characterize the emerging behavior through targetted insight rather than naive brute *simulation.* On the one hand, it relies on an understanding of how our system is affected when changes are incurred to it. On the other hand, it relies, inversely, on understanding what changes can be effected into the system to sustain or induce a desired behavior.

The work becomes actionable through a good use of the link connecting the behavior of the complicated systems to its separate subsystems through the machinery of exact sequences. **We will term this connection, for the rest of this section, as the *linking sequence.*** Some examples include:

- **Forward problems.** The theory allows us to deduce the behavior of the interconnected system from the extracts of the subsystems, through the linking sequence, without need of recomputation. The theory thus enables us to update the behavior of a system when small local changes are made to it, bypassing full computations.

  Example. Given that we know the image and kernel of a matrix A, updating the kernel once B is appended to A becomes easy. The alternative consists of recomputing the kernel from the big matrix.

- **Inverse problems.** The theory allows us to derive criteria from the linking sequence that characterizes the subsystems that ought to be added to a system so as to produce, or preserve, a desired phenomena. Every system whose extract satisfies the criterion would be a valid candidate.

  Example. The characterization automatically gives us a criterion to understand the class of matrices B that yield, when appended, a kernel of a given dimension. The alternative consists of defining that class by trying out all possibilities.

- **Divide and conquer.** A complicated system may be divided into simpler interacting systems. Each of the simple subsystems are easy to understand, yet understanding their interconnection is daunting. The linking sequence would enable to bypass daunting computations.

  Example. If C can be split into simpler matrices A and B, the accessible information on A and B can then be combined to yield the information for the big matrix.

As the extracts further tend to be smaller than the systems, it provides an even augmented computational benefit. Intricate use of the linking sequence should further provide theoretical enhancements when understanding systems and their interaction.

1.4.3 The linking side.

The nature of the theory allows a unification of disparaged instances of the phenomenon. The value of the conceptual and actionable contribution is vindicated by the library of models and classes of systems it can account for. The thesis visits several models and instances, including:

- Cascading phenomena through the lens of the behavioral approach.
- Contagion phenomena as depicted by networked models.
- Reachability problems, cascading failures and paths synchronization.
- Interconnection and the effects of memory.
- Problems in *interconnecting* matrices.

The linking sequence is directly obtained in the case where the problem has a linear nature. Most problem will not possess that nature, and thus a solution is arrived to through appropriate lifts. The thesis provides such lifts, for various classes of systems, based on group rings, abelian sheaves and abelian complexes.

The theory establishes links to logic, algebraic topology and algebraic geometry among other areas in mathematics. The aim is to open up a fertile perspective in understanding interconnected systems and the phenomena that emerge.

1.5 Background and Genesis of the work.

This work began as an endeavor to understand cascade effects. It precisely first aimed at understanding the mathematical structure underlying models of diffusion of behavior commonly studied in the social sciences. The setup there consists of a population of interacting agents. In a societal setting, the agents may refer to individuals. The interaction of the agents affect their behaviors or opinions. The goal is to understand the spread of a certain behavior among agents given certain interaction patterns. Threshold models of behaviors (captured by M.0, M.1, M.2 and M.3 in Chapter 4) have appeared in the work of Granovetter [Gra78], and more recently in [Mor00]. Such models are key models in the literature, and have been later considered by computer scientists, see. e.g., [Kle07] for an overview.

Chapter 4 develops a class of systems abstracting away those models. A system in such a class interpreted as a collection of (monotone) implications: if such and such event occurs, then such event occurs. From an elementary viewpoint, those systems consist of a state space, a partially ordered set $(P, \leq)$, and some dynamics equipped over the state space. We show in Chapter 4 that each system may be identified with a map $f : P \to P$ satifying the following three axioms:

A.1 If $a \in P$, then $a \leq f(a)$.

A.2 If $a \leq b$, then $f(a) \leq f(b)$.

A.3 If $a \in P$, then $f(f(a)) = f(a)$.

Every such map conversely yields a potential system. Such a class of systems encapsulates two features. The first feature embodies the seemingly consequential effects that emerge from such mathematical models. These systems capture chain of events. The second feature enables a theory of interaction and combination. We gain a notion of subsystem, and a notion of interconnection that coincides with the act of combining descriptions.

Those maps are often known as *closure operators*. On one end, they appeared in the work of Tarski (see e.g., [Tar36] and [Tar56]) to formalize the notion of deduction. On another end, they appeared in the work of Birkhoff, Ore and Ward (see e.g., [Bir36], [Ore43] and [War42], respectively), parts of foundational work in universal algebra. The first origin reflects the consequential relation in the effects considered. The second origin reflects the theory of interaction of multiple systems. Closure operators appear as early as [Moo10].

The ideas of the thesis are salient in this class of systems. First, there had to be a clear separation between the system itself (the semantics) and a representation of it (the syntax). Fundamental properties of the systems ought to be representation independent. Second, interconnection of systems had to be consistent with the interconnection at the syntactic level. Our systems were described by text (or implicational statements) and not wiring diagrams. Thus input/output composition is not the solution, and the mathematical structure of interconnection is differemt. Interconnections instead consists of taking joins in a lattice formed by those maps. We had gained a notion of subsystem, and a notion of interconnection that coincides with the act of combining descriptions.

These ideas resonate well with Jan C. Willems' behavioral approach to systems theory. Indeed, the theory of interaction later on builds on the intuition provided by the behavioral approach. Looking closely into the three axioms of the class reveals A.2, that of order-preserveness, to be the most essential. This axiom A.2 may be replaced by *Scott-continuity* (see e.g., [Sco72] for a definition), and leads us straight into D. Scott's work on denotational semantics and domain theory (see e.g., [Sco71], [SS71] and [Sco72]). From this view point, the emphasis on the separation between syntax and semantics ought to be greater, and brings us closer to many ideas in logic and formal methods. The path of considering systems satisfying A.2 (or Scott-continuity) would lead us to algebras of system akin to those introduced by combinatory logic. Scott's domains have also been models for the λ-calculus. This suggests developing interesting formal languages for the systems in play. The λ-calculus has also been given semantics through cartesian closed categories. The general pattern of the behavioral approach exhibited in Chapter 5 opens ways for links with such languages. This direction has been set aside for the sake of another. But there many connections to study in the interplay of syntax and semantics.

There is still however a major problem in understanding cascading phenomena. We have an algebra of systems that deals with interconnection, but what gives these systems their potential for cascade effects? What is the mathematical feature that amounts to the intuition of cascading phenomena? Interconnecting two systems only yields an interconnected system. Also, the lattice of systems up to this point could easily be replaced by any another lattice. We thus view them arise when we decide to forget something from the system. Of course, the structure of the systems should at least give us something to forget. And thus the mathematical feature may eventually be found in the structure of the lattice itself.

As an example, let us consider N. Chomsky's theory of generative grammar. Every grammar builds one language, and different grammars may describe the same language. But the grammars generating the same language are different. Adding a clause to one could yield a very different effect than adding it to another. Similar things happen in deduction, and later on in other problems pertaining cascade effects. How do those grammars however gain this *generativity*? Well it is coming from the grammar rules. But how do we capture it? We capture it by destroying the rules, and studying how the grammar without the rules (amounting to only the language) behaves when combined with other grammars. It is the vivid discrepancy in the outcome between the presence of the rules and their abscence that encodes the generativity. Thus to capture cascade effects resulting from the interaction of systems we perform the following experiment. On one end, we let the systems interact and observe the outcome. On another end, we destroy the interdependencies lying in the systems and let them interact without them. These two ends, in the presence of cascading phenomena, will show a discrepancy in outcome. This discreprancy then encodes the phenomenon. Studying the discrepancy amounts to studying the phenomenon.

This outlook brings out the following scheme:

$$\langle P, + \rangle \xleftarrow{\Phi} \langle S, \oplus \rangle$$

The space S is the space of systems, and the space P is the space of *phenomes*, what we observe from the systems. The map Φ is termed the veil. It conceals (or forgets) things from a system and keeps only the phenome visible. Thus there are two ingredients that are required. The first ingredient is a theory of interconnection or interaction. The second ingredient is theory of higher order effects, obtained by forgetting things from the systems. The effects would then emerge whenever:

$$\Phi(s \oplus s') \neq \Phi(s) + \Phi(s)$$

The schematic is roughly finalized, but it is still by no means clear what properties should the mathematical spaces and arrows have. We need those two ingredients to be synergetic, as they are to live in the basin of a same theory. One natural approach is to think of an algebra of systems along the lines of universal algebra. The presence of the operators requires some amount of structure on the spaces, but that ought not cause a problem. However homomorphisms are designed to be structure preserving. But interesting phenomena arise when the arrow between the spaces is not a homomorphism, but one that exactly fails to keep the structure. A different perspective is then sought.

A closer look in the case of lattices reveals a few things. First, an arrow Φ is not considered to be a lattice homomorphism, but only an order homomorphism. It is then the order relation that is essential, and not the join (interaction) operation. Second, having all systems be subsystems of some universe systems (as in the case of lattices) seemed restrictive. Replacing order-relations with morphisms and minimums by universal arrows, left us a trail to consider categories in general. On one end, the generalized notion of interaction/interconnection though colimits fitted perfectly. On another end, the free construction of algebraic objects seemed a prominent example of something exhibiting cascading phenomena. The notion of adjunction itself seemed to have captured the essence of what was required. In case of preorders, it restricts to Galois connections.

At this point, these cascade effects arise from a loss of exactness caused by a right adjoint functor. If our categories of systems and phenomes were abelian categories, then the veil is an additive left exact functor. The potential of a veil to destroy exactness can then be measured. We can then compute cohomology objects from the systems, giving rise to the derived functors of the veil. As interconnection of systems can then be made to coincide with exact sequences, we can use those cohomology objects to recover the lost exactness, due to generativity, through a long exact sequence.

1.6 Scope and outline.

This body of work has been architected from one growing intuition. It has however been split into different self-contained pieces, each expounding an aspect of it at a certain technical level. Each chapter is one piece. The chapters tend to be self-contained, and the technical mathematical background required varies, intentionally, from one to the other.

1.6.1 Chapter 2 – Generativity and interactional effects: an overview.

This chapter provides an overview of the theory. This chapter proposes a means to relate properties of an interconnected system to its separate component systems in the presence of cascade-like phenomena. Building on a theory of interconnection reminiscent of the behavioral approach to system theory, this chapter introduces the notion of *generativity*, and its byproduct, *generative effects*. Cascade effects, enclosing contagion phenomena and cascading failures, are seen as instances of generative effects. The latter are precisely the instances where properties of interest are not preserved or behave very badly when systems interact. The goal of the chapter is to overcome that obstruction. We will show how to extract mathematical objects from the systems, that encode their generativity: their potential to generate new phenomena upon interaction. Those objects may then be used to link the properties of the interconnected system to its separate systems. Such a link will be executed through the use of exact sequences from commutative algebra.

The theory of generativity may be developed on two levels. The first is a special level where the systems in play tend to be all subsystems of a fixed system serving as a universe. The second is a general level where such a universe system is not present. We will only be concerned with the special level in this chapter. It is much simpler to describe. One however needs the general level to insightfully arrive at a clear formulation of the problem, and then at a solution. This chapter can then only present a sketch of the solution, rather than the general details of it. This chapter will nevertheless illustrate a working example solution in the special level.

1.6.2 Chapter 3 – Where do cascades come from?

We argue that the mathematical structure enabling cascade-like effects to intuitively emerge coincides with certain Galois connections. We introduce the notion of generative effects to formalize cascade-like phenomena. We define the notion of a veil, and show that such effects arise from either concealing mechanisms or forgetting characteristic from a system. We study properties of the veil, introduce dynamical veils and end by discussing factorizations and lifts. The goal is to initiate a mathematical base that enables us to further study such phenomena. In particular, generative effects will be linked to a certain loss of exactness. Homological algebra, and related algebraic methods, then come in to characterize such effects.

1.6.3 Chapter 4 – Towards an algebra for cascade effects.

We introduce a new class of (dynamical) systems that inherently capture cascading effects (viewed as consequential effects) and are naturally amenable to combinations. We develop an axiomatic general theory around those systems, and guide the endeavor towards an understanding of cascading failure. The theory evolves as an interplay of lattices and fixed points, and its results may be instantiated to commonly studied *models* of cascade effects.

We characterize the systems through their fixed points, and equip them with two operators. We uncover properties of the operators, and express *global* systems through combinations of *local* systems. We enhance the theory with a notion of failure, and understand the class of shocks inducing a system to failure. We develop a notion of μ-rank to capture the energy of a system, and understand the minimal amount of *effort* required to fail a system, termed *resilience.* We deduce a dual notion of *fragility* and show that the combination of systems sets a limit on the amount of fragility inherited.

1.6.4 Chapter 5 – On the abstract structure of the behavioral approach.

We revisit the behavioral approach to systems theory and make explicit the abstract pattern that governs it. Our end goal is to use that pattern to understand interaction-related phenomena that emerge when systems interact. Rather than thinking of a system as a pair $(\mathbb{U}, \mathcal{B})$, we begin by thinking of it as an injective map $\mathcal{B} \to \mathbb{U}$. This relative perspective naturally brings about the sought structure, which we summarize in three points. First, the separation of behavioral equations and behavior is developed through two spaces, one of syntax and another of semantics, linked by an interpretation map. Second, the notion of interconnection and variable sharing is shown to be a construction of the same nature as that of gluing topological spaces or taking amalgamated sums of algebraic objects. Third, the notion of interconnection instantiates to both the syntax space and the semantics space, and the interpretation map is shown to preserve the interconnection when going from syntax to semantics. This pattern, in its generality, is made precise by borrowing very basic constructs from the language of categories and functors.

1.6.5 Chapter 6 – Interconnection and memory in LTI systems.

We characterize the role played by memory when linear time-invariant systems interact. This study is of interest as the phenomenon that occurs in this setting is arguably the same phenomenon that governs cascading failure and contagion effects in interconnected systems. We aim to later extend solutions presented in this chapter to problems in other desired settings.

The characterization relies on basic methods in homological algebra, and is reminiscent of the rank-nullity theorem of linear algebra. Interconnection of systems is first expressed as an exact sequence, then loss of memory causes a loss of exactness, and finally exactness is recovered through specific algebraic invariants of the systems that encode the role of memory. We thus introduce a new invariant, termed *lag*, of linear time-invariant systems and characterize the role of memory in terms of the lag. We discuss properties of the lag, and prove several results regarding the characterization.

1.6.6 Chapter 7 – Cascading phenomena in the behavioral approach.

This chapter studies the behavior of a subsystem as parts of its greater system undergo changes. As changes can lead, by means of interconnections, to changes in *remote* subsystems, the situation is inherently one that exhibits cascade-like effects. We cast the situation through the lens of the behavioral approach to systems theory, and recover a characterization relating the behavior of the subsystem to that of its greater system and the incurred change. We develop a short general theory to address the posed situation, and instantiate it to five cases: linear finite-dimensional systems, affine systems, finite systems, linear time-invariant systems and systems defined by polynomial equations. The theory relies on methods from homological algebra, and uncovers the *zero-dynamics* of a system as essential to relate the behavior of a subsystem to its greater system. The general pattern exhibited by the theory is of separate interest to understand interaction-related phenomena that generally occur in the interaction of systems.

1.6.7 Chapter 8 – Generativity and interactional effects: general theory.

This chapter develops the theory at the general functorial level. The chapter exposes the emergence of interaction-related phenomena as a loss of exactness. It introduces the notion of generativity, and its by-product generative effects. These occur precisely when properties or features of a system behave badly under interconnection. The chapter outlines, develops and exemplifies homological methods to deal with such phenomena. The goal is to relate the behavior of the interconnected system to that of its separate components despite the presence of such phenomena.

1.6.8 Chapter 9 – How to make cascade effects linear.

It can be a common (mis)conception that cascading phenomena arise from *non-linearities*. The key message expressed throughout the thesis is that they do not. The mathematical structure underlying cascades is loss of exactness. As such it is then conceivable to lift our problematic situation to a world that is linear, keeping the effects intact. In such linear (or *abelian*) settings, tools from commutative algebra and homological algebra can be put to good use in understanding the phenomena, notably through defining (co)homology theories. We introduce the notion of an (abelian) veil-lift to encode the phenomenon (somewhat) intact in an abelian structure. We develops tools and tricks to abelianize cascading phenomena, and finally show that every situation admits an abelian veil-lift. The neverending goal is then to find tight lifts.

1.6.9 Chapter 10 – Conclusion.

We conclude the thesis with some remarks and future directions.

Chapter 2

Generativity and interactional effects: an overview

Abstract

This chapter proposes a means to relate properties of an interconnected system to its separate component systems in the presence of cascade-like phenomena. Building on a theory of interconnection reminiscent of the behavioral approach to system theory, this chapter introduces the notion of *generativity*, and its byproduct, *generative effects*. Cascade effects, enclosing contagion phenomena and cascading failures, are seen as instances of generative effects. The latter are precisely the instances where properties of interest are not preserved or behave very badly when systems interact. The goal of the chapter is to overcome that obstruction. We will show how to extract mathematical objects from the systems, that encode their generativity: their potential to generate new phenomena upon interaction. Those objects may then be used to link the properties of the interconnected system to its separate systems. Such a link will be executed through the use of exact sequences from commutative algebra.

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2.1 Introduction.

Whenever we deal with the realm of interconnection of systems and interaction-related effects—termed, *interactional effects*—we may be driven by a natural impulse to desire properties of systems, explaining such effects, that are compositional. By compositional properties, we mean properties that are preserved or behave very well when the systems are interconnected. Interactional effects of heavy interest, such as contagion effects and cascading failures, arise however exactly because compositionality fails in various aspects. These phenomena are given away by systemic properties that fundamentally do not behave well under interconnection. We may then fairly expect, when aiming to understand such interactional effects, any non-trivial tangible compositional property to be fundamentally too weak to yield us something of use. But if we back away from the idea of wanting our properties to be compositional, can we recover such compositionality through other means? This chapter proposes such a means to relate properties of an interconnected system to its separate component systems in the presence of cascade-like phenomena. Properties that are preserved or behave well under interconnection are seen to be related through a special case of that means.

This chapter introduces the notion of *generativity*, and its byproduct, *generative effects*. Cascade effects, enclosing contagion phenomena and cascading failures, are seen as instances of generative effects. The converse does not have to be true unless one decides to expand the intuition that floats about cascading phenomena. The key to understanding these effects is that they are **not** intrinsic to the system. They rather result from an extrinsic dichotomy, a separation between what is deemed observable in the system—termed, the *phenome*—and what is concealed in the system. Generative effects emerge from an interplay between the phenome and the concealed mechanisms. This dichotomy is enforced by a map—termed, the *veil*—from a space of systems to a space of phenomes. The veil partially covers the system leaving the phenome bare, thereby concealing mechanisms in a system. Generative effects are sustained by the veil whenever the phenome of the interconnected system cannot be explained by the phenome of the separate parts. The mechanisms of the systems concealed under the veil interact so as to produce new observables.

Generative effects are thus precisely the instances where the property of interest, the phenome, evolves horrendously under interconnection. The goal of the chapter is to overcome that obstruction. We will show how to extract mathematical objects from the systems, that encode their generativity: their potential to generate new observables. Those objects may then be used to link the phenome of the interconnected system to its separate systems. Such a link will be executed through the use of exact sequences from commutative algebra. The horizon goal of such a development is twofold. We firstly aim to acquire a computational means to evaluate the behavior of systems under the presence of cascading-like phenomena. We secondly aim to gain insight into how systems interact and behave among each others. Such insight is to be be used first for the (theoretical) analysis of interactive systems, and second for the design of systems that can desirably cope with change.

The theory of generativity may be developed on two levels. The first is a special level where the systems in play tend to be all subsystems of a fixed system serving as a universe. The second is a general level where such a universe system is not present. We will only be concerned with the special level in this chapter. It is much simpler to describe. One however needs the general level to insightfully arrive at a clear formulation of the problem, and then at a solution. This chapter can then only present a sketch of the solution, rather than the general details of it. This chapter will nevertheless illustrate a working example solution in the special level.

This chapter begins by a theory of interconnection reminiscent of the behavioral approach to system theory, initiated by J. C. Willems. It proceeds to define generative effects, and elaborates examples to develop the intuition of the reader. In a linear world, generative effects will be associated to a loss of surjectivity. We thus show how to recover such a loss, and link the phenome of the interconnected system to the separate component systems. As not all worlds are linear, we arrive at a general solution by lifting our problems to linear problems.

On the technical end, interconnection of systems, in a linear world, will be synonymous to short exact sequences. Generative effects are sustained by a veil if, and only if, exactness is lost once the veil is applied to the systems. The goal is then to recover this loss of exactness. We can extract algebraic objects from the systems, that encode their generativity, and mend the nonexact sequence into a long exact one by fitting in the objects appropriately.

2.2 Interconnection and interaction of systems.

We cannot have interactional effects without a notion of interaction or interconnection. The notion of interconnection in the special theory is rather simple, yet inclusive. The systems are elements of a set S, and interaction is an operation $\vee : S \times S \to S$ that is:

I.1. Associative i.e., $(s \vee s') \vee s'' = s \vee (s' \vee s'')$

I.2. Commutative i.e., $s \vee s' = s' \vee s$

I.3. Idempotent i.e., $s \vee s = s$

A semilattice $(S, \vee)$ is a set S equipped with an operation $\vee$ that is associative, commutative and idempotent. The operation $\vee$ is termed the *join* of the semilattice.

Definition 2.2.1. *A system is an element of a semilattice* $(S, \vee)$*. The systems* s *and* s' *in* S *are interconnected to give the system* $s \vee s'$*, their join, in* S*.*

A system s is defined to be a subsystem of s' whenever $s \vee s' = s'$. The system $s = s_1 \vee \cdots \vee s_m$ is then the system that amounts from the interaction, or combination, of the subsystems $s_1, \cdots, s_m$. There is only a unique way to interconnect two systems, and it is via the $\vee$ operation.

A semilattice $(S, \vee)$ induces a partial order $\leq$ on S obtained by setting $s \leq s'$ if, and only if, $s \vee s' = s'$. Thus s is a subsystem of s' if, and only if, $s \leq s'$. The join $s \vee s'$ is then the least upper bound of s and s'. It is the smallest system t such that $s \leq t$ and $s' \leq t$.

We provide two generic concrete interpretations, and leave other interpretations to examples in future sections.

2.2.1 The behavioral approach to system theory.

The behavioral approach to system theory, initiated by J. C. Willems, begins with the premise that a mathematical model acts as an exclusion law (see e.g., [PW98] and [Wil07]). The phenomenon we wish to model produces events or outcomes that live in a given set $\mathbb{U}$, termed the *universum*. The laws of the model (viewed descriptively) state that some outcomes in $\mathbb{U}$ are possible, while others are not. The model then restricts the outcomes in $\mathbb{U}$ to only those are allowed possible by the laws of the model. The set of possible outcomes is then called the *behavior* of a model. We will refrain from using the term model, and replace it by the term system.

Definition 2.2.2 (cf. [PW98] Section 1.2.1)**.** *A Willems system is a pair* $(\mathbb{U}, \mathcal{B})$ *with* $\mathbb{U}$ *a set, called the* universum*—its elements are called* outcomes*—and* $\mathcal{B}$ *a subset of* $\mathbb{U}$ *called the* behavior*.*

In case we fix a universum $\mathbb{U}$, the set of Willems systems $(\mathbb{U}, \mathcal{B})$ partially ordered as $(\mathbb{U}, \mathcal{B}) \leq (\mathbb{U}, \mathcal{B}')$ if, and only if, $\mathcal{B} \supseteq \mathcal{B}'$ forms a semilattice $(S, \vee)$. Interconnection of systems is given by the set-intersection of the behaviors, and corresponds to the join of the defined semilattice.

Proposition 2.2.3. *If* $(\mathbb{U}, \mathcal{B})$ *and* $(\mathbb{U}, \mathcal{B}')$ *are Willems systems, then their interconnection* $(\mathbb{U}, \mathcal{B} \cap \mathcal{B}')$ *is given by* $(\mathbb{U}, \mathcal{B}) \vee (\mathbb{U}, \mathcal{B}')$*.*

The properties of the semilattice will depend on what we allow as possible behaviors. If we consider all subsets to be possible behaviors, our semilattice $(S, \vee)$ will form a Boolean lattice. If $\mathbb{U}$ is a vector space, and we consider the linear subspaces of $\mathbb{U}$ to be the possible behaviors, then we tend to get a semilattice that is only a modular lattice.

2.2.2 Syntactical systems and descriptions.

Another approach consists of thinking of an element of the semilattice $(D, \vee)$ as a description of a system. Descriptions are combined through the join operation of the semilattice. The description may be in the form of a text, an equation, a diagram or any syntactical piece one might wish for. Inasmuch as the solution set of a set of algebraic equations does not depend on their order, we have $d \vee d' = d' \vee d$. Inasmuch as redundant algebraic equations produce no effects on the solution set, we have $d \vee d = d$. As combining descriptions also tends to be associative, we arrive at the defining axioms of the semilattice.

As a simplified formalization. Let Σ be a finite set, termed *the alphabet*. We define Σ^* to be the set of finite strings, words, or sequences, made up of elements of Σ. If $\Sigma = \{a, b\}$, then $\Sigma^* = \{\emptyset, a, b, aa, ab, ba, bb, aaa, aab, \cdots\}$. A description, termed *language*, is a subset of Σ^*. As not all languages may provide meaningful descriptions for our systems, we may a pick a subset $\mathcal{L}$ of them. We order $\mathcal{L}$ by inclusion, and get a partial order. We will assume that every pair of languages in $\mathcal{L}$ admits a least upper-bound. We would have then obtained a semilattice $(\mathcal{L}, \vee)$. If d and d' are descriptions of systems in $\mathcal{L}$, then $d \vee d'$ denotes the smallest language in $\mathcal{L}$ that contains both d and d'.

If d and d' are descriptions for the same system, then we expect $d \vee d'$ to be a redundant description of the same system. Let us assume that $\mathcal{L}$ is finite. If a system admits multiple descriptions of it in $\mathcal{L}$, we may take the join of all those descriptions to arrive at the maximum language in $\mathcal{L}$ that describes the system. The collection of such maximal languages forms a subsemilattice $(S, \vee)$ of $\mathcal{L}$. The semilattice S will then be our semilattice of systems. In case every system admits a unique description in $\mathcal{L}$, the semilattice of systems S is just $\mathcal{L}$.

2.3 Generativity and Interactional Effects.

A theory of interconnection by itself will not be enough to produce interactional effects. Interconnecting two systems only gives an interconnected system. We thus view interactional effects as fundamentally not intrinsic to the system. They will only emerge once we set our expectation for what is deemed observable in a system.

Let **System** be a semilattice of systems. We define the phenome as that which we choose to explicitly observe from an arbitrary system in that class. A phenome may be either a property, a feature, a consequence, or even a subsystem of the system. We generally, often non-trivially, arrive at a phenome by forgetting irrelevant information from the system. We may lightly define a phenome as the image of a system under a map $\phi : \textbf{System} \to P$ of sets. We may choose to forget nothing at all, and get the identical whole system as a phenome. The set P would then be **System**, and ϕ would be the identity map. We may also choose to forget everything, and get *nothing* as a phenome of a system. The set P would then be a singleton set $\{*\}$, and ϕ would be the unique map $\textbf{System} \to \{*\}$. Thus varying what and how much we forget from a generic system of **System** gives us different phenomes for the same system.

We are interested in understanding how the phenome of a certain system changes when the system is modified. More generally, we want to understand how the phenome of systems changes when systems interact. We are particularly interested in the situations where the phenome of the interconnected system cannot be explained by the phenome of the separate systems. Once a phenome is declared, everything we intentionally forget from the system is declared to be concealed. Although concealed features of a system may not emerge by themselves into the phenome, they are likely to interact with phenomes or concealed mechanisms from other systems to affect the observable phenome. Such situations are characteristic of the contagion behavior observed in societal settings and of cascading failure in various infrastructural systems. The parts of the systems that are declared concealed may interact so as to produce more than what is expected from what is observable. We term the effects leading to such unexplained phenomena as *generative effects*. The concealed part of the system is irrelevant to the system's phenome, but it has the potential to interact with either the phenome or concealed parts of other systems. We term that potential *generativity*.

2.3.1 Veils and generativity.

We arrive at a phenome by forgetting things from a system, by concealing them under a veil. We may then well think of a phenome as a simplified system. The set of phenomes then forms a semilattice $(P, \vee)$. The join $\vee$ naturally induces a partial order $\leq$ on P.

Definition 2.3.1 (Veil)**.** *A veil on* **System** *is a pair (P, ϕ) where $(P, \vee)$ is a semilattice of phenomes, and $\phi : \textbf{System} \to P$ is a map such that:*

V.1. The map ϕ is order-preserving, i.e., if $s \leq s'$, then $\phi s \leq \phi s'$.

V.2. Every phenome admits a simplest system that explains it, i.e., the set $\{s : p \leq \phi s\}$ has a (unique) minimum element for every phenome p.

The veil is intended to hide away parts of the system, and leave other parts, the phenome, of the system bare and observable. The axiom V.1 indicates that concealing a subsystem of a system may only yield a subphenome of the phoneme of the system. The axiom V.2 indicates that everything one observes can be completed in a simplest way to something that extends under the veil. Generative effects occur precisely when one fails to explain the happenings through the observable part of the system. In those settings, the things concealed under the veil would have interacted and produced observable phenomes.

Definition 2.3.2 (Generative Effects)**.** *A veil (P, ϕ) is said to sustain generative effects if $\phi(s \vee s') \neq \phi(s) \vee \phi(s')$ for some s and s'.*

Different veils may be chosen for the same semilattice of systems. Some will sustain generative effects and some will not. For instance, both veils $(\textbf{System}, id)$ and $(\{*\}, * : \textbf{System} \to \{*\})$ do not sustain generative effects at all. All that can be observed is explained by what is already observed. Thus the standard intuition for systems exhibiting cascading phenomena, or contagion effects, does not stem from a property of a system. It is rather the case that the situation admits a highly suggestive phenome and highly suggestive veil that sustains such effects. Those effects are thus properties of the situation. Should we change the veil, we may either increase those effects, diminish them or even make them completely go away. Such interactional effects depend only on what we wish to observe.

Our aim is then to answer the following question:

Question 2.3.3. *Given a veil (P, ϕ) that sustains generative effects, how can we non-trivially characterize or express $\Phi(s \vee s')$ through separate information on s and s' (and potentially a common system $s \wedge s'$)?*

How can we relate the behavior of the interconnected system to its separate components?

2.4 Some examples.

The aim of this section is to develop the reader's intuition on veils and generative effects. We provide five examples.

2.4.1 Generativity is not intrinsic to the system.

This example deals with a very simple—if not the simplest—instance of generative effects. Let U be a finite set. Two proper subsets of U are not equal to U by definition. Their union can however be equal to U. If we set up a veil that keeps only whether a given subset of U is equal to U or not, the veil will then sustain generative effects. Formally, let $\{*\}$ be a one point set. The semilattice **System** of systems is $(2^U, \cup)$. The veil is $(2^{\{*\}}, \phi)$ where $\phi S = \emptyset$ if $S \subsetneq U$ and $\phi S = \{*\}$ if $S = U$. Although two subsets are not separately equal to U, their union can be U. If S^c is the complement set of $S \subsetneq U$, then:

$$\phi(S \cup S^c) \neq \phi S \cup \phi S^c$$

This toy instance may be complicated by replacing 2^U by any semilattice L. Let $s, s' \in L$ be non-comparable elements, and define ϕ to map an element t to $\{*\}$ if $t \geq s \vee s'$, and to $\emptyset$ otherwise. Such a defined veil trivially sustains generative effects for any semilattice of systems with two non-comparable elements. It is then ill-posed to talk about a system exhibiting generative effects. It is a property of the perspective, i.e. the veil, we choose.

2.4.2 Generativity in the behavioral approach.

Let us consider a mega-system comprised of an interacting mixture of infrastructures (e.g., power, transportation, communication), markets (e.g., prices, firms, consumers), political entities and many individuals. We are interested in understanding the evolution of the behavior of a subsystem of this mega-system, as changes are effected into the mega-system. Of courses changes directly effected onto the subsystem modifies the behavior. It is also the case that seemingly non-related changes causes a shift in the behavior by a successive chain of events.

Let $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{R}$ be sets such that $\mathbb{M} = \mathbb{S} \times \mathbb{R}$. Following the behavioral approach terminology we will have $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{R}$ be the outcome space, or universum, of the **m**ega-system, the **s**ubsystem, and the **r**est (or remainder) in the mega-system that is not the subsystem of interest. The systems will then be subsets of those universa. Specifically, the sets $\mathcal{M} \subseteq \mathbb{M}$, $\mathcal{S} \subseteq \mathbb{S}$ and $\mathcal{R} \subseteq \mathbb{R}$ denote the behavior of the mega-system, the subsystem and the rest, respectively. Although $\mathcal{S}$ is a subsystem of $\mathcal{M}$, the set $\mathcal{S}$ is not a subset of $\mathcal{M}$, but is rather a projection (or a quotient) of $\mathcal{M}$ onto the $\mathbb{S}$-coordinate. A change in our mega-system, following the behavioral approach, is then depicted as an intersection with a change $\mathcal{C} \subset \mathbb{M}$. If we denote by $\pi : \mathbb{M} \to \mathbb{S}$ the projection onto the $\mathbb{S}$-coordinate, then $\pi\mathcal{M} = \mathcal{S}$ and we generally observe:

$$\pi(\mathcal{M} \cap \mathcal{C}) \neq \pi(\mathcal{M}) \cap \pi(\mathcal{C}).$$

The change $\mathcal{C}$ affects the subsystem $\mathcal{S}$ through interactions within $\mathcal{R}$. If all the interactions were confined to be within $\mathcal{R}$, then we would have had equality for sure. In such a case, changes outside of $\mathcal{S}$ do not affect $\mathcal{S}$.

Mathematically.

The systems lattice is $(2^{\mathbb{M}}, \cap)$, and the lattice of phenomes is then $(2^{\mathbb{S}}, \cap)$. The veil π is then the projection of $\mathcal{M} \subseteq \mathbb{M}$ onto the $\mathbb{S}$-coordinate, i.e.,

$$\pi\mathcal{M} = \{s \in \mathbb{S} : (s, r) \in \mathcal{M}\}.$$

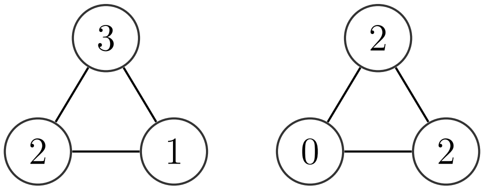
The map π preserves the partial order, and every phenome $\mathcal{S} \subseteq \mathbb{S}$ admits $\mathcal{S} \times \mathbb{R}$ as a simplest system explaining it in $2^{\mathbb{M}}$. The veil π also sustains generative effects. As a simple instance, consider $\mathbb{S} = \{s, \boldsymbol{s}\}$ and $\mathbb{R} = \{r, \boldsymbol{r}\}$. Let our mega-system be $\mathcal{M} = \{(s, r), (\boldsymbol{s}, \boldsymbol{r})\}$, where all feasible outcomes have matched type-faces. Our system $\mathcal{S}$ is then $\pi\mathcal{M} = \{s, \boldsymbol{s}\}$. We will now effect the following change $\mathcal{C} = \{(s, \boldsymbol{r}), (\boldsymbol{s}, \boldsymbol{r})\}$, where only bold-faced $\boldsymbol{r}$ is allowed. We then observe an inequality. The set $\pi(\mathcal{M} \cap \mathcal{C}) = \{\boldsymbol{s}\}$ is different than the set $\pi(\mathcal{M}) \cap \pi(\mathcal{C}) = \{s, \boldsymbol{s}\}$. The change *propagated* through $\mathcal{R}$ into the behavior of $\mathcal{S}$.

Generally.

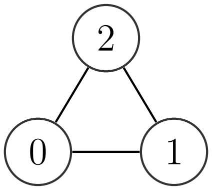
The universa $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{R}$ may be equipped with various mathematical structures, e.g., linear structures making them vector space. The behaviors become subspaces of their corresponding universum. This example may then be enriched as needed.

2.4.3 Deduction and consequences.

As a simplified case of this example, we will have each system consist of a three node graph. Each node in the graph can be colored either black or white, and is assigned an integer k as a threshold. All nodes are white initially. A node then becomes black, if at least k of its neighbors are black. Once a node is black it remains black forever. In this setting, the order of update does not affect the final set of black. For instance, let A and B denote the systems on the left and right, respectively.



Given our rule above, a threshold of 0 indicates that a node *automatically* becomes black. If no threshold of 0 exists, then necessarily all nodes will remain white. Two systems interact by combining their evolution rules. The system $A \vee B$ corresponds to the graph that keeps on each node the minimum threshold between that of A and B:



We can forget the evolution rules that are prone to interact with others by keeping from the systems only the set of final black nodes. Indeed, every set of black nodes S corresponds to a simplified system having a threshold of 0 on the nodes in S and a threshold of ∞ on the nodes not in S. Let us denote by $\phi(A)$ and $\phi(B)$ the set of black nodes of A and B respectively. Then the set $\phi(A)$ is empty, and the set $\phi(B)$ contains the left node. The combination of the phenomes $\phi(A)$ and $\phi(B)$ corresponds to the union $\phi(A) \cup \phi(B)$. Such a combination may be equivalently thought of as the final set of black nodes corresponding to $\phi(A)$ and $\phi(B)$ when viewed as simplified systems. We then arrive at the inequality:

$$\phi(A \vee B) \neq \phi(A) \cup \phi(B)$$

When A and B are combined, the left black node in B interacts with the rules of A to color the right node black. Both the left and the right nodes then interact with the rules of B to color the middle node black. This effect is encoded in the inequality.

Generally.

The above case may be trivially generalized to arbitrary graphs and thresholds. The essence of it however lies in the following idea. Informally, let $E_1, \cdots, E_n$ be statements. A statement may be thought of as expression that may either be proven or unproven to be true. It is helpful to think of statements as theorems. The statements however may be related. If some are proved to be true, they may constitute a proof for other statements to be true. The systems will then consist of a set of premises and implications. The premises may be thought of as axioms, and the implications may be thought of as inference rules. Some statements are initially assumed true, and the implications allow us to prove more statements to be true than initially held. Two systems are combined by combining their premises and the implication rules. It is intuitively clear that the interaction of implications from the systems would allows a far more powerful deduction than what is possible without interaction.

Mathematically.

The systems and their properties are treated in [Ada17c]. Let S be a set. The systems can be identified with maps $f : 2^S \to 2^S$ satisfying:

A.1 For all $A \subseteq S$, we have $A \subseteq fA$.

A.2 If $A \subseteq B$, then $fA \subseteq fB$.

A.3 For all $A \subseteq S$, we have $ffA = fA$.

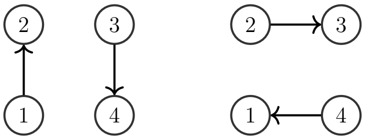
We can order the maps by $f \leq g$ if, and only if, $fA \leq gA$ for all $A \subseteq S$. We then obtain a semilattice $\mathcal{L}$. If we define $\phi : \mathcal{L} \to 2^S$ to map f to its least fixed-point $f(\emptyset)$, then $(2^S, \phi)$ defines a veil. This veil can be shown to sustain generative effects.

The lattice 2^S may also be replace by any other lattice. Of course, one may also consider subsemilattices of $\mathcal{L}$ for additional variations.

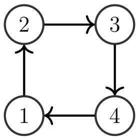
2.4.4 Reachability problems.

This example may viewed through the lens of reachability problems. A system loosely consists of a collection of states along with internal evolution dynamics. The dynamics of system dictate whether the system may evolve from state a to state b. Two systems may be combined by allowing their dynamics to interact. The interaction of dynamics would then allow the system to reach more states than what is separately reachable.

As a simplified case of this example, we will have each system consist of a digraph over four nodes. For instance, let S and S' denote the systems on the left and right, respectively.



Two systems S and S' interact by combining their edges, to yield $S \cup S'$.



The phenome of the system corresponds to the set of pairs $a \to b$ where b can be reached from a through a directed path. We can ignore the cases $a \to a$ as they belong to the phenome of every possible system. Then phenome $\phi(S)$ of S corresponds to $\{1 \to 2, 3 \to 4\}$, while the phenome $\phi(S')$ of S' corresponds to $\{2 \to 3, 4 \to 1\}$. The phenomes $\phi(S)$ and $\phi(S')$ are combined via set union to yield $\phi(S) \cup \phi(S')$. One can then see that:

$$\phi(S \cup S') \neq \phi(S) \cup \phi(S')$$

Generative effects are sustained. Indeed $\phi(S) \cup \phi(S')$ contains only four elements, whereas $\phi(S \cup S')$ contains all possible pairs. The edges in the combined graph are aligned to create paths that did not exist separately.

Generally + Mathematically.

Let S be a set. Denote by $\mathbf{Rel}(S)$ and $\mathbf{Tran}(S)$ the set of relations on S and transitive relations on S, respectively. The set $\mathbf{Rel}(S)$ froms a semilattice by defining the join to be union of sets, and $\mathbf{Tran}(S)$ forms a semilattice by defining the join ot be the union of sets followed by the transitive closure. The systems lattice is then $\mathbf{Tran}(S)$, the phenomes lattice is $\mathbf{Rel}(S)$ and our veil will be defined to forget the transitive property. The map ϕ of the veil will be the order-preserving inclusion from $\mathbf{Tran}(S)$ to $\mathbf{Rel}(S)$. The defined veil sustains generative effects.

Important remark.

The semilattice of phenomes does not have to be *smaller* than the semilattice of systems. The veil can be devised to forget properties, and thus the phenome consists of systems that do not necessarily have the forgotten property. The space of phenomes then trivially contains the systems that do have the forgotten property. Throwing away information from the system, leaves us with a system with less information. But if that information was restrictive, then the space of phenomes will be *greater* than that of the systems.

2.4.5 Words, languages and grammars.

Let $\Sigma = \{a, b\}$ be an alphabet set. A word over Σ is a string consisting of a finite sequence of letters in Σ, e.g., *abba*, *a*, *abaab*, etc. A system will consist of a collection of transformation rules $u \leftrightarrow v$ where u and v are words. Starting from a given word w, a system that possesses rule $u \leftrightarrow v$ may substitute any appearance of u as a subword in w by a subword v, and vice versa. Two systems are combined by taking the union of the rules. Fixing an initial word w, the phenome we are interested in is the set of words that a system may transform it to. Indeed, the scope may be much greater when systems are combined than what can be separately achieved. Generative effects will be sustained.

As a concrete instance, let the system S be the rules $aa \leftrightarrow a$ and $bb \leftrightarrow b$, and the system S' be the rules $ab \leftrightarrow ba$. Let us pick (and fix) w to be ab. The phenome $\phi(S)$

is then the set of words having all as on the left and all bs on the right. The phenome $\phi(S')$ is only the set $\{ab, ba\}$.

As two systems interact by putting their relations in common, the system $S \vee S'$ is the rules $aa \leftrightarrow a$, $bb \leftrightarrow b$ and $ab \leftrightarrow ba$. The combination of phenomes is the set union $\phi(S) \cup \phi(S')$. Generative effects are sustained as:

$$\phi(S \vee S') \neq \phi(S) \cup \phi(S').$$

Indeed, the phenome $S \vee S'$ contains all strings containing at least one a and one b.

Mathematically.

This example falls within the world of languages and grammars. We will unfortunately neglect the algebraic structure of the problem, for the purposes of this example. Let Σ^* be the set of all words, and let w be a fixed word. Every system corresponds to an equivalence relation on Σ^*. The phenome corresponds to the equivalence class containing the word w. Combination of systems corresponds to closing the union relation under transitivity, and combination of phenomes is set union. Both the systems and the phenomes form semilattices. The veil that reads the equivalence class of w can be shown to sustain generative effects.

2.5 The problem and the goal.

Our aim is to understand the evolution of the phenome as systems interact. The inequality in generative effects hinders such an understanding. Since we are precisely interested in such effects, we are bound to live with that inequality. The question then becomes as to how we go around it. We are interested in the phenome of the interconnected system. We may obviously, if tractable, combine the systems and read the phenome. Such an approach, however, will yield no insight at all into the problem.

The inequality of generative effects tells us that some features of the combined phenome cannot be explained by the separate ones. Thus we still need to extract additional information from the system. We will then extract a mathematical object that encodes the generativity of a system: the potential of a system to produce changes in the phenome. We can then use these objects to relate the phenome of the combined system to that of the separate subsystems. Thus, those object will summarize the required information needed to go around the inequality. But most importantly, in most cases, the system cannot be reconstructed from the objects and the phenomes. We are then distilling what it is that makes systems produce those effects. In the general theory, these objects may be seen as universal in a certain sense. The development in this chapter will however be oblivious to any property those objects ought to possess.

The key to the solution is that the inequality in generative effects means that some features of the combined system's phenome cannot be *explained* by the separate subsystems' phenome.

2.5.1 Destroying surjectivity.

A veil (P, ϕ) is said to sustain generative effects if $\phi(s \vee s') \neq \phi(s) \vee \phi(s')$ for some s and s'. The phenome of the separate systems is, thus, unable to explain the phenome of the interconnected system. This inability will be formally understood as a loss of surjectivity of a certain map. The loss of surjectivity will be key to the solution.

Let us suppose that both the systems and the phenomes are sets. More precisely, we will have $\mathbf{System} = (2^U, \cup)$ and $P = (2^V, \cup)$. If S and S' are subsets of U, then there are canonical injective maps $i : S \to S \cup S'$ and $i' : S' \to S \cup S'$. Let $S \amalg S'$ denote the disjoint union of S and S', and define $i \amalg i' : S \amalg S' \to S \cup S'$ to be the map i on S and i' and S'.

Proposition 2.5.1. *For every every S and S', the map $i \amalg i'$ is a surjective map.*

Proof. An element of $S \cup S'$ belongs to either S, S' or both. □

On another end, as $\phi S \subset \phi(S \cup S')$ and $\phi S' \subset \phi(S \cup S')$, we get canonical injective maps $\phi i : \phi S \to \phi(S \cup S')$, and $\phi i' : \phi S' \to \phi(S \cup S')$. The map $\phi i \amalg \phi i'$ need not always be surjective.

Proposition 2.5.2. *The veil ϕ sustain generative effects if, and only if, the map $\phi i \amalg \phi i' : \phi S \amalg \phi S' \to \phi(S \cup S')$ is not surjective for some S and S'.*

Proof. Generative effects are not sustained if, and only if, $\phi(S \cup S') = \phi(S) \cup \phi(S')$ for all S and S'. If $\phi(S \cup S') = \phi(S) \cup \phi(S')$, then $\phi i \amalg \phi i' : \phi S \amalg \phi S' \to \phi(S \cup S')$ is surjective by Proposition 2.5.1. Conversely, if $\phi i \amalg \phi i'$ is not surjective, then some element in $\phi(S \cup S')$ does not admit a preimage in $\phi S \amalg \phi S'$. Therefore, $\phi(S \cup S')$ strictly contains $\phi(S) \cup \phi(S')$, and generative effects are sustained. □

Generative effects then occur when there are points in $\phi(S \cup S')$ that do not admit preimages in either ϕS or $\phi S'$.

2.6 The problem, in a world that is linear.

We may push the findings further if we equip our setting with more structure. We will have both the systems and the phenomes be vector spaces. If V is a vector spaces, we define $\mathbf{Sub}(V)$ to be the lattice of the subspaces of V. The join of A and B in $\mathbf{Sub}(V)$ is $A + B$, the linear span of A and B. Let V and W be vector spaces, we consider a veil $\phi : \mathbf{Sub}(V) \to \mathbf{Sub}(W)$.

As a concrete example, one may consider $\phi : \mathbf{Sub}(\mathbb{R}^n) \to \mathbf{Sub}(\mathbb{R}^{n-1})$ obtained by intersecting a subspace of $\mathbb{R}^n$ by a fixed hyperplane H. One may check that $(\mathbf{Sub}(\mathbb{R}^{n-1}), \phi)$ is a veil for $\mathbf{Sub}(\mathbb{R}^n)$ that sustains generative effects for every hyperplane.

If S and S' are subspaces of V, we then have two injective linear maps $i : S \to S + S'$ and $i' : S' \to S + S'$. We can then form a linear map $i - i' : S \oplus S' \to S + S'$.

Proposition 2.6.1. *For every S and S', the map $i - i'$ is a surjective map.*

Proof. Every element of $S + S'$ can be written in the form $a - (-a')$ with $a \in S$ and $-a' \in S'$. □

On another end, as $\phi S \subset \phi(S + S')$, and $\phi S' \subset \phi(S + S')$ we get linear maps $\phi i : \phi S \to \phi(S + S')$, and $\phi i' : \phi S' \to \phi(S + S')$. The map $\phi i - \phi i'$ need not always be surjective.

Proposition 2.6.2. *The veil ϕ sustain generative effects if, and only if, the map $\phi i - \phi i' : \phi S \oplus \phi S' \to \phi(S + S')$ is not surjective for some S and S'.*

Proof. The same reasoning as that in the proof of Proposition 2.5.2 applies, with $\cup$ replaced by $+$. □

If $\phi i - \phi i'$ is not surjective, then there are elements in $\phi(S + S')$ that do not admit a preimage in either ϕS or $\phi S'$. Those points cannot be explained by ϕS and $\phi S'$.

In the linear case, we win an extra characterization of what cannot be explained by the phenome. If I denote the image of $\phi i - \phi i'$, then $\phi(S + S') = I \oplus \phi(S + S')/I$. What can be explained by the phenome lies in I. What cannot be explained by the phenome, and is caused by generative effects, lies in $\phi(S + S')/I$. Our goal is to characterize and recover $\phi(S + S')/I$.

2.6.1 Interlude on exact sequences.

If we live in a linear world, the relationship among the phenomes of the interconnected system and its separate subsystems will be established through the use of exact sequences.

A sequence of $\mathbb{R}$-vector spaces V_i and linear maps f_i

$$\cdots \longrightarrow V_{i-1} \xrightarrow{f_i} V_i \xrightarrow{f_{i+1}} V_{i+1} \longrightarrow \cdots$$

is said to be exact at V_i if $\operatorname{im} f_i = \ker f_{i+1}$. The sequence is said to be exact if it is exact at every V_i. In particular, the sequence:

$$0 \longrightarrow U \xrightarrow{f} V$$

is exact if, and only if, the map f is injective. Dually, the sequence:

$$V \xrightarrow{g} W \longrightarrow 0$$

is exact if, and only if, the map f is surjective. As we shall see, if the phenomes $\phi(S)$, $\phi(S')$ and $\phi(S + S')$ were made to be part of an exact sequences, we can then relate them together. For example, if the sequence:

$$U \xrightarrow{f} V \xrightarrow{g} W$$

is exact, then V is isomorphic to $\operatorname{im} f \oplus \ker g$. If the sequence is longer, the characterization may reach elements further apart.

2.6.2 Loss of exactness, on the right.

Let V and W be vector spaces. We again consider **System** to be $\mathbf{Sub}(V)$ and set up a veil $(\mathbf{Sub}(W), \phi)$. For S and S' subspaces in V, let $j : S \to S+S'$ and $j' : S' \to S+S'$ be the canonical injections. The sequence:

$$S \oplus S' \xrightarrow{j-j'} S + S' \longrightarrow 0$$

is always exact. Generative effects occur precisely when:

$$\phi S \oplus \phi S' \xrightarrow{\phi j - \phi j'} \phi(S + S') \longrightarrow 0$$

is not exact. There is then a non-zero vector space U corresponding to $\operatorname{coker}(\phi j - \phi j')$ and a surjective map $\phi(S + S') \to U$ such that the sequence:

$$\phi S \oplus \phi S' \xrightarrow{\phi j - \phi j'} \phi(S + S') \longrightarrow U \longrightarrow 0$$

is exact. The vector space U corresponds to the unexplained phenomes, and is isomorphic to $\operatorname{coker}(\phi j - \phi j')$. We, of course, do not know the map $\phi j - \phi j'$ as we do not know the phenome $\phi(S + S')$. The goal is then to recover U from information on the systems S and S'. As an exemplary approach, we may perform such a recovery via the Snake lemma.

Proposition 2.6.3 (Snake Lemma, e.g., [AM69] ch. 2, p. 23, proposition 2.10)**.** *Given a commutative diagram of vector spaces with exact rows,*

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "f"] \arrow[d, "u"] & V \arrow[r, "g"] \arrow[d, "v"] & W \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & U \arrow[r, "f'"] & V' \arrow[r, "g'"] & W' \arrow[r] & 0
\end{tikzcd}$$

we get an exact sequence:

$$0 \longrightarrow \ker u \xrightarrow{\tilde{f}} \ker v \xrightarrow{\tilde{g}} \ker w \xrightarrow{\delta} \operatorname{coker} u \xrightarrow{\bar{f'}} \operatorname{coker} v \xrightarrow{\bar{g'}} \operatorname{coker} w \longrightarrow 0.$$

Proof. The lemma is standard, and its proof may be found in many texts, e.g., [AM69] ch. 2, p. 22. □

The exact sequence derived form the Snake lemma allows us to link the kernel and cokernel of w to those of u and v.

Proposition 2.6.4. *If the sequence of vector spaces:*

$$0 \longrightarrow V_0 \xrightarrow{f} V_1 \longrightarrow V_2 \longrightarrow V_3 \xrightarrow{g} V_4 \longrightarrow V_5 \longrightarrow 0,$$

is exact, then $V_2 = \operatorname{coker} f \oplus \ker g$ and $V_5 = \operatorname{coker} g$.

Proof. The following sequence is exact:

$$0 \to \operatorname{im}(V_1 \to V_2) \to V_2 \to \operatorname{im}(V_2 \to V_3) \to 0$$

We have $\operatorname{im}(V_1 \to V_2) = V_1/\ker(V_1 \to V_2)$. As $\ker(V_1 \to V_2) = \operatorname{im} f$ by exactness of the six-term sequence, we get that $\operatorname{im}(V_1 \to V_2) = \operatorname{coker}(f)$. By exactness, we also get $\operatorname{im}(V_2 \to V_3) = \ker g$. Finally, short exact sequence of vectors spaces split. Namely, if $0 \to U \to V \to W \to 0$ is a sequence of vector spaces, then $V = U \oplus W$. □

The strategy to recover the phenome coming from generative effects would be to lift the sequence $S \oplus S' \to S + S' \to 0$ to be part of the diagram described in the Snake lemma. If we can think of our systems as linear maps, and encode the phenome as a kernel of those maps, then we may recover the map $\phi S \oplus \phi S' \to \phi(S+S')$ as part of the kernel-cokernel exact sequence. Furthermore, three columns are in play in the Snake lemma. The middle would correspond to the separate systems. The rightmost column would correspond to the interconnected system. The leftmost will be made to correspond to the common part of the two systems on which they will be interconnected.

As a general insight, let V and V' be subspaces of some vector space, and consider the following commutative diagram:

$$\begin{tikzcd}
V \cap V' \arrow[r, "i"] \arrow[d, "i'"] & V \arrow[d, "j"] \\
V' \arrow[r, "j'"] & V + V'
\end{tikzcd}$$

Proposition 2.6.5. *The sequence*

$$0 \longrightarrow V \cap V' \xrightarrow{(i,i')} V \oplus V' \xrightarrow{j-j'} V + V' \longrightarrow 0$$

is exact.

Proof. The map (i, i') is injective and $j - j'$ is surjective. The map $j - j'$ maps every element (v, v) to 0. Conversely, if $j(v) - j'(v') = 0$, then $j(v) = j'(v')$. As j and j' are injective, we get $v = v'$ and thus $v \in V \cap V'$. □

Interconnection of systems can then be thought of as a short exact sequence. Generative effects is then equivalent to a loss of exactness, but only on the right.

Proposition 2.6.6. *The veil ϕ sustains generative effects, if and only if, the sequence*

$$0 \longrightarrow \phi(S \cap S') \xrightarrow{(\phi i, \phi i')} \phi S \oplus \phi S' \xrightarrow{\phi j - \phi j'} \phi(S + S') \longrightarrow 0$$

is not exact at $\phi(S + S')$ for some S and S'. The sequence is always exact at both $\phi(S \cap S')$ and $\phi S \oplus \phi S'$.

Proof. The map $(\phi i, \phi i')$ is injective, thus the sequence is always exact at $\phi(S \cap S')$. The map $\phi j - \phi j'$ maps every element (v, v) to 0. Conversely, if $\phi j(v) - \phi j'(v') = 0$, then $\phi j(v) = \phi j'(v')$ and so $v = v'$ and belongs to $\phi(S) \cap \phi(S')$. As ϕ is a veil, we have $\phi(S \cap S') = \phi(S) \cap \phi(S')$. The loss of exactness on the right is characterized by Proposition 2.6.2. □

Through the use of the Snake lemma, we may then relate the phenome of the interconnected system to the separate subsystems and their common part. Of course knowing the phenomes of the separate systems and their common part does not entail us to know the phenome of the interconnected systems. The cokernels in the exact sequence hold the additional information required to deduce generated phenomes. The cokernels will encode the generativity of the systems.

2.7 The solution, in a world that is not linear.

We may directly apply the above technique only if the systems are *linear*. However, most of the settings were are interested in do not possess a linear structure. The goal is to lift our problems, say, to vector spaces. Such a lift may be for instance achieved by encoding the desired information in the dimension of a vector space.

We will develop in this section a characterization for a simple formulation as a model example. The formulation is the common ground for most of the examples given in Section 2.4.

2.7.1 The formulation.

Let S be a set $\{e_1, \cdots, e_n\}$ of n elements. Given an undirected graph G over S, we are interested in whether or not there is an undirected path from e_1 to e_n in G. Neither of the following graphs G or G' contains a path from e_1 to e_3.

$$G: \qquad a \text{ --- } b \qquad c \qquad\qquad G': \qquad a \qquad b \text{ --- } c$$

But if G and G' are combined together, by taking the union of their edge set,

$$G \cup G': \qquad a \text{ --- } b \text{ --- } c$$

the edges synchronize and a path emerges.

Let $\mathcal{G}$ denote the semilattice of undirected graphs over S, where the join of $G, G' \in \mathcal{G}$ is the graph $G \cup G'$ containing the union of the separate edges. We define $\phi : \mathcal{G} \to 2^{\{*\}}$ to be the map such that $\phi G = \{*\}$ if e_1 and e_n are connected in G and $\phi G = \emptyset$ otherwise.

The map ϕ is not yet a veil, as it does not satisfy V.2. We can make ϕ to be a veil by restricting it to only graphs that are a disjoint union of cliques. Those graphs form a semilattice where the join consists of combining the edges first, then adding edges in each existing connected component to form cliques. We may then define a system to be a disjoint union of cliques, or equivalently, an equivalence relation on S. The fix will, however, not affect the solution at all. The lift we perform will treat both the graph and its *closure* as the same. We may then just ignore the fix, and consider all undirected graphs over S as systems.

2.7.2 The lift.

To lift our problem into a linear world, we let $\mathbb{R}^S$ denote the free vector space with basis $\{e_1, \cdots, e_n\}$. If $G \in \mathcal{G}$, we define a subspace I_G of $\mathbb{R}^S$ to be the span of the vectors $e_i - e_j$ where $\{e_i, e_j\}$ is an edge in G. We then lift every G to a map:

$$g : \mathbb{R}^2 \to \mathbb{R}^S / I_G$$

obtained by the composition of the inclusion $\mathbb{R}^2 \to \mathbb{R}^S$ that sends the generators of $\mathbb{R}^2$ to e_1 and e_n in $\mathbb{R}^S$, and the canonical surjection $\mathbb{R}^S \to \mathbb{R}^S/I_G$.

Proposition 2.7.1. *The dimension of $\mathbb{R}^S/I_G$ is equal to the number of connected components in G.* □

The kernel of the map, denoted by $\Phi(I_G)$, will then encode the phenome. For every G, we know that e_1 and e_n are not in I_G. Furthermore, $e_1 - e_n \in I_G$ if, and only if, e_1 and e_n are connected in G via a path. We then have:

Proposition 2.7.2. *The kernel $\Phi(I_G)$ is isomorphic to $\mathbb{R}$ if e_1 and e_n are connected in G and is the 0 vector space otherwise.* □

The lift also preserves interconnection of systems.

Proposition 2.7.3. *If $G, G' \in \mathcal{G}$, then $I_{G \cup G'} = I_G + I_{G'}$.* □

In general, the space $I_G \cap I'_G$ is non-necessarily isomorphic to $I_{G \cap G'}$, even if G and G' are disjoint unions of cliques. Finally, an inclusion of graphs induces maps on the lifts:

Proposition 2.7.4. *Let H be a subgraph of G, the inclusion graph homomorphism $H \to G$ lifts to a commutative diagram:*

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "id"] \arrow[d, "h"] & \mathbb{R}^2 \arrow[d, "g"] \\
\mathbb{R}^S/I_H \arrow[r, "i"] & \mathbb{R}^S/I_G
\end{tikzcd}$$

where i is the canonical linear map.

Proof. Let $h : H \to G$ be an inclusion graph homomorphism. If $\{i, j\}$ is an edge in H, then $\{h(i), h(j)\}$ is an edge in G. Thus if $i - j \in I_H$, then $hi - hj \in I_G$. Therefore $I_H \simeq hI_H \subseteq I_G$. The canonical surjection $\mathbb{R}^S \to \mathbb{R}^S/I_G$ then factors through $\mathbb{R}^S/I_H$ to yield i. □

The connected components of a graph G form a basis for $\mathbb{R}^S/I_G$. As H is a subgraph of G, the map $i : \mathbb{R}^S/I_H \to \mathbb{R}^S/I_G$ is surjective and sends connected components of H to connected components of G in a manner compatible with the inclusion.

2.7.3 Recovering exactness.

Those square diagrams can then be neatly fitted in an exact diagram:

Proposition 2.7.5. *The following diagram is commutative and has exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^2 \arrow[r] \arrow[d] & \mathbb{R}^2 \oplus \mathbb{R}^2 \arrow[r] \arrow[d, "(g{,}g')"'] & \mathbb{R}^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R}^S/(I_G \cap I_{G'}) \arrow[r, "(i{,}i')"] & \mathbb{R}^S/I_G \oplus \mathbb{R}^S/I_{G'} \arrow[r, "j-j'"] & \mathbb{R}^S/(I_G + I_{G'}) \arrow[r] & 0
\end{tikzcd}$$

Proof. To show exactness of the bottom row, apply the Snake lemma to the canonical diagram:

$$\begin{tikzcd}
0 \arrow[r] & I_G \cap I_{G'} \arrow[r] \arrow[d] & I_G \oplus I_{G'} \arrow[r] \arrow[d] & I_G + I_{G'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R}^S \arrow[r] & \mathbb{R}^S \oplus \mathbb{R}^S \arrow[r] & \mathbb{R}^S \arrow[r] & 0
\end{tikzcd}$$

whose upper row we know is exact from Proposition 2.6.5. □

We may then recover an exact sequence from the Snake lemma. We first summarize the pieces of the sequence. Every square diagram:

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "id"] \arrow[d, "h"] & \mathbb{R}^2 \arrow[d, "g"] \\
\mathbb{R}^S/I_H \arrow[r, "i"] & \mathbb{R}^S/I_G
\end{tikzcd}$$

can be extended to a commutative diagram with exact columns:

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\Phi(I_H) \arrow[r, "\Phi(i)"] \arrow[d] & & \Phi(I_G) \arrow[d] \\
\mathbb{R}^2 \arrow[r, "id"] \arrow[d, "h"] & & \mathbb{R}^2 \arrow[d, "g"] \\
\mathbb{R}^S/I_H \arrow[r, "i"] \arrow[d] & & \mathbb{R}^S/I_G \arrow[d] \\
\mathbb{H}(I_H) = \mathbb{R}^S/(I_H + \langle e_1, e_n \rangle) \arrow[rr, "\mathbb{H}(i)"] \arrow[d] & & \mathbb{H}(I_G) = \mathbb{R}^S/(I_G + \langle e_1, e_n \rangle) \arrow[d] \\
0 & & 0
\end{tikzcd}$$

The space $\langle e_1, e_n\rangle$ is the subspace of $\mathbb{R}^S$ generated by e_1 and e_n. The vector space $\mathbb{R}^S/(I_G+\langle e_1, e_n\rangle)$, denoted by $\mathbb{H}(I_G)$, is the cokernel of g, and the map $\mathbb{H}(i) : \mathbb{H}(I_H) \to \mathbb{H}(I_G)$ sends an element $a + I_H + \langle e_1, e_n\rangle$ in $\mathbb{H}(I_H)$ to $i(a + W_H) + \langle e_1, e_n\rangle$ in $\mathbb{H}(I_G)$.

The dimension of the space $\mathbb{H}(I_G)$ is equal to the number of connected components in G not containing either e_1 or e_n. The map $\mathbb{H}(i)$ then destroys all the components in H that do not contain e_1 or e_n in H but that do contain them in the image component.

Finally, if $G, G' \in \mathcal{G}$, we recover an exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & \Phi(I_G \cap I_{G'}) \arrow[r, "{(\Phi(i),\Phi(i'))}"] & \Phi I_G \oplus \Phi I_{G'} \arrow[r, "\Phi(j)-\Phi(j')"] & \Phi(I_{G\cup G'}) \arrow[dll] & \\
 & \mathbb{H}(I_G \cap I_{G'}) \arrow[r, "{(\mathbb{H}(i),\mathbb{H}(i'))}"'] & \mathbb{H}I_G \oplus \mathbb{H}I_{G'} \arrow[r, "\mathbb{H}(j)-\mathbb{H}(j')"'] & \mathbb{H}(I_{G\cup G'}) \arrow[r] & 0
\end{tikzcd}$$

Proposition 2.7.6. *We have:*

$$\Phi(I_{G\cup G'}) = \operatorname{coker}((\Phi(i), \Phi(i'))) \oplus \ker((\mathbb{H}(i), \mathbb{H}(i')))$$

and:

$$\mathbb{H}(I_{G\cup G'}) = \operatorname{coker}((\mathbb{H}(i), \mathbb{H}(i'))).$$

Proof. Apply Proposition 2.6.4 to the six-term exact sequence. □

The space $\operatorname{coker}((\Phi(i), \Phi(i')))$ encodes whether or not e_1 and e_n are connected in one of the separate graphs, and the space $\ker((\mathbb{H}(i), \mathbb{H}(i')))$, or equivalently $\ker(\mathbb{H}(i))\cap \ker(\mathbb{H}(i'))$, encodes the formation of such a path via generative effects. In particular,

Proposition 2.7.7. *If* $\Phi(I_G) = \Phi(I_{G'}) = 0$, *then* $\Phi(I_{G\cup G'}) = \ker(\mathbb{H}(i)) \cap \ker(\mathbb{H}(i'))$. □

As an explicit characterization, we have:

$$\ker((\mathbb{H}(i), \mathbb{H}(i'))) = {}^{(I_G + \langle e_1, e_n\rangle) \cap (I_{G'} + \langle e_1, e_n\rangle)}\big/_{I_G \cap I_{G'} + \langle e_1, e_n\rangle}$$

Whether or not a path is created is then encoded in the difference of the dimensions of $(I_G + \langle e_1, e_n\rangle) \cap (I_{G'} + \langle e_1, e_n\rangle)$ and $I_G \cap I_{G'} + \langle e_1, e_n\rangle$. Such a discrepancy would exist as the lattice of subspaces in a not a distributive lattice, but only a modular one. This remark will not be further pursued. The explicit characterization provided will however be further considered in a later section dealing with graphs in multiple universa.

The vector spaces $\mathbb{H}(I_.)$ also admit additional structure that allows them to be interpreted combinatorially.

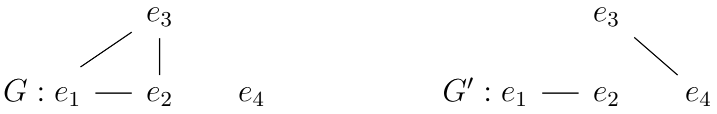
2.7.4 A combinatorial criterion.

The graphs, in this section, are assumed, for simplicity, to be disjoint union of cliques. Let $\mathcal{M}(G) \subseteq 2^S$ be the collection of non-empty subsets $V \subseteq S$ of vertices, where the induced subgraph of G on V has a perfect matching. The vector space I_G can be derived from the set $\mathcal{M}(G)$. The vector space $I_G \cap I_{G'}$ can be derived from $\mathcal{M}(G) \cap \mathcal{M}(G')$.

Definition 2.7.8. *If $\mathcal{M} \subseteq 2^S$, we define:*

$$\mathbb{H}\mathcal{M} = \Big\{\{e_1\}, \{e_n\}\Big\} \cup \Big\{V - \{e_1, e_n\} : V \in \mathcal{M} \textit{ is minimal}\Big\}.$$

The operation $\mathbb{H}$ *splits* the minimal elements of $\mathcal{M}$ whenever they contain either e_1 or e_n. The set $\mathbb{H}\mathcal{M}(G)$ then corresponds to the induced subgraph of G on the nodes not connected to e_1 or e_n. The splitting procedure, thus, destroys information in $\mathcal{M}$. For instance, if G and G' are the graphs:



then $\mathbb{H}\mathcal{M}(G) = \mathbb{H}\mathcal{M}(G') = \big\{\{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}\big\}$, although $\mathcal{M}(G)$ and $\mathcal{M}(G')$ are different.

Proposition 2.7.9. *If $\Phi(I_G) = \Phi(I_{G'}) = 0$, then $\Phi(I_{G \cup G'}) = \mathbb{R}$ if, and only if, there exists a set $V \subseteq S$ such that:*

C.1. The set V is a disjoint union of sets in $\mathbb{H}\mathcal{M}(G)$,

C.1'. The set V is a disjoint union of sets in $\mathbb{H}\mathcal{M}(G')$,

C.2. The set V is not a disjoint union of sets in $\mathbb{H}(\mathcal{M}(G) \cap \mathcal{M}(G'))$.

Proof. Let V be a minimal such set. As C.1 (resp. C.1') holds, the nodes in V that are not connected to either e_1 or e_n in G (resp. G') can be perfectly matched in G (resp. G'). As C.2 holds, by minimality of V, no two nodes in V share the same component in both G' and G. Furthermore, as C.2 holds, no subset in V can belong to a cycle in $G \cup G'$. The nodes in V then have to form a path between e_1 and e_n in $G \cup G'$.

Conversely, if $\Phi(I_{G \cup G'}) = \mathbb{R}$, then $\ker \mathbb{H}(i)$ and $\ker \mathbb{H}(i')$ have a common element, an alternating sum $\sum_k (-1)^k e_{i_k}$ for $e_{i_k} \in S$ a vertex and $0 \leq k \leq m-1$. Pick the common element consisting of the least number of vertices. This set of vertices forms a set V satisfying C.1, C.1' and C.2. $\square$

2.7.5 Concrete instances.

The answer to the following three questions is obviously yes. We answer them to illustrate the workings of the above theory.

Q.1. If we combine G and H, do we get a path from a to c?

$$G: \qquad a — b \qquad c \qquad\qquad H: \qquad a \qquad b — c$$

We have $I_G = \langle a{-}b \rangle$ and $I_H = \langle b{-}c \rangle$, and thus $I_G \cap I_H = 0$. We then get $\mathbb{H}(i) : \mathbb{R} \to 0$ and $\mathbb{H}(i') : \mathbb{R} \to 0$. Clearly then $\Phi(I_{G\cup G}) = \ker(\mathbb{H}(i)) \cap \ker(\mathbb{H}(i')) = \mathbb{R}$.

Combinatorially, we have:

$$\mathbb{H}\mathcal{M}(G) = \mathbb{H}\mathcal{M}(H) = \big\{\{a\}, \{b\}, \{c\}\big\}.$$

The set $\mathbb{H}\mathcal{M}(G) \cap \mathcal{M}(H)$ is $\big\{\{a\}.\{c\}\big\}$ as no subset supports a perfect matching in both graphs. The set $\{b\}$ satisfies the required conditions.

Q.2. If we combine G and H, do we get a path from a to e?

$$G: \qquad a — b \qquad c — d \qquad e \qquad\qquad H: \qquad a \qquad b — c \qquad d — e$$

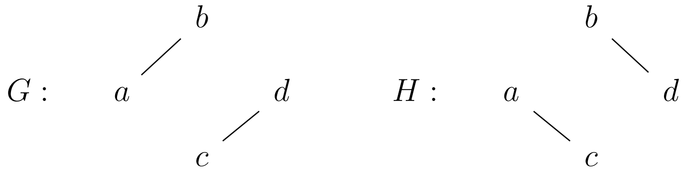
We have $I_G = \langle a - b, d - c \rangle$ and $I_H = \langle b - c, d - e \rangle$, and thus $I_G \cap I_H = 0$. We get $\mathbb{H}(i) : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbb{H}(i') : \mathbb{R}^3 \to \mathbb{R}$ where $\mathbb{R}^3$ is generated by b, c and d. The map $\mathbb{H}(i)$ sends both b and $c - d$ to 0. The map $\mathbb{H}(i')$ sends both d and $b - c$ to 0. The element $b - c + d$ then generates the intersection of the kernels. We then get $\Phi(I_{G\cup H}) = \mathbb{R}$.

Combinatorially, we have:

$$\mathbb{H}\mathcal{M}(G) = \big\{\{a\}, \{b\}, \{c, d\}, \{e\}\big\} \quad \textit{and} \quad \mathbb{H}\mathcal{M}(H) = \big\{\{a\}, \{b, c\}, \{d\}, \{e\}\big\}.$$

The set $\mathbb{H}\mathcal{M}(G) \cap \mathcal{M}(H)$ is $\big\{\{a\}.\{e\}\big\}$ as no subset supports a perfect matching in both graphs. The set $\{b, c, d\}$ satisfies the required conditions.

Q.3. If we combine G and H, do we get a path from a to d?



We have $I_G = \langle a{-}b, c{-}d \rangle$ and $I_H = \langle a{-}c, b{-}d \rangle$, and then $I_G \cap I_H = \langle a{-}b{+}d{-}c \rangle$. We get $\mathbb{H}(i) : \mathbb{R} \to 0$ and $\mathbb{H}(i') : \mathbb{R} \to 0$. Clearly then $\Phi(I_{G\cup G}) = \ker(\mathbb{H}(i)) \cap \ker(\mathbb{H}(i')) = \mathbb{R}$.

Combinatorially, we have:

$$\mathbb{H}\mathcal{M}(G) = \mathbb{H}\mathcal{M}(H) = \big\{\{a\}, \{b\}, \{c\}, \{d\}\big\}.$$

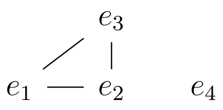
The set $\mathbb{H}\mathcal{M}(G) \cap \mathcal{M}(H)$ is $\big\{\{a\}, \{b, c\}, \{d\}\big\}$. Both the sets $\{b\}$ and $\{c\}$ satisfy the required conditions.

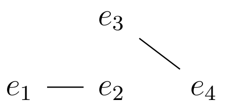
2.7.6 Encoding generativity.

We have thus related the phenome of the combined graph to the separate graphs through the objects $\mathbb{H}(I)$ via an exact sequence:

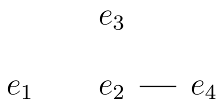
$$\begin{tikzcd}
0 \arrow[r] & \Phi(I_G \cap I_{G'}) \arrow[r, "{(\Phi(i),\Phi(i'))}"] & \Phi(I_G) \oplus \Phi(I_{G'}) \arrow[r, "\Phi(j)-\Phi(j')"] & \Phi(I_{G\cup G'}) \arrow[dll] & \\
 & \mathbb{H}(I_G \cap I_{G'}) \arrow[r, "{(\mathbb{H}(i),\mathbb{H}(i'))}"'] & \mathbb{H}(I_G) \oplus \mathbb{H}(I_{G'}) \arrow[r, "\mathbb{H}(j)-\mathbb{H}(j')"'] & \mathbb{H}(I_{G\cup G'}) \arrow[r] & 0
\end{tikzcd}$$

The objects $\mathbb{H}(I_{\cdot})$ may be then be seen to encode at least what is essential explain generative effects. As such, if the objects $\mathbb{H}(I_{\cdot})$ are always 0, one then concludes that generative effects are not sustained. As a rough converse, the objects $\mathbb{H}(I_G)$ may be seen to encode only what is essential for generative effects in the system. Indeed, we did not use all the information of the separate graphs. For instance, the problem of combining any of the two graphs:





with the graph:



will yield the same exact sequence from the Snake Lemma. The special theory developed in this chapter is not, however, set up to discuss well how much of the information is kept from the system. One however ought to expect a good variation. In some cases, a large amount of information is irrelevant and will be brushed away by the $\mathbb{H}(I)$ objects. On another end, one may devise example where almost everything from the system is fundamentally bound to play a part in generating effects. Such an example may go along the lines of Example 2.4.1. In such situations, the generativity of the system tends to get close to exactly what is concealed under the veil.

2.7.7 For graphs living on different vertices.

The graphs G and G' have both been defined over the same vertex set S. The same problem can be more generally recast by gluing two graphs defined over different vertices over a common subsets of vertices. The problem is formally set up as follows.

Let G and G' be undirected graphs over V and V' respectively, and let C be a set. The set C is to be interpreted as the set of common vertices. As such we are given inclusions:

$$i : C \to V \quad \text{and} \quad i' : C \to V'.$$

The nodes $i(v) \in V$ and $i'(v) \in V'$ will be identified as the same vertices, to form the glued graph. The gluing construction mathematically amounts to taking a pushout from C along i and i'. Pushouts however are outside the scope of this chapter, and we thus revert to an algorithmic construction. We form an undirected graph G^* over a set V^*. If n, n' and c denote the cardinality of V, V' and C, then V^* has cardinality $n + n' - c$. The set V^* will then be seen to contain both V and V'. As such we have two inclusions:

$$j : V \to V^* \quad \text{and} \quad j' : V' \to V^*,$$

whose images coincide on C. The edges of G^* are $j(u) \sim j(v)$ whenever $u \sim v$ in G and $j'(u') \sim j'(v')$ whenever $u' \sim v'$ in G'.

We then pick two distinguished vertices s and t. Each of the vertices s and t may either lie in V, in V' or in both (i.e., in C). The question then is:

Question 2.7.10. *Given G and G' to be glued along a vertex set C, and a compatible choice of s and t, is there an undirected path from s to t in G^*?*

If the set C is equal to V and V' (i.e., i and i' are bijections), we then recover the original formulation of the situation, whereby the graphs are defined over the same vertex set.

The characterization.

We get different cases, depending on whether or not s and t lie in V or V'. All the cases can be dealt with mathematically in an implicit manner. We will however deal with some explicitely for clarity of exposition, and present the rest in a generic form. We however omit proofs and refer the reader to [Ada17g] for more details.

Let n, n' and c be the cardinality of V, V' and C. The inclusion $i : C \to V$ (resp. $i : C \to V'$) induces an injective linear map $\iota : \mathbb{R}^c \to \mathbb{R}^n$ (resp. $\iota' : \mathbb{R}^c \to \mathbb{R}^{n'}$). If I is a subspace of $\mathbb{R}^n$ (resp. of $\mathbb{R}^{n'}$), we define $\pi(I)$ (resp. $\pi'(I)$) to be $\{a \in R^c : \iota a \in I\}$ (resp. $\pi'(I) = \{a \in R^c : \iota' a \in I\}$. Note that $\pi(I)$ and $\pi'(I)$ are both subspaces of $\mathbb{R}^c$.

We will suppose that no path from s to t already exists in G or in G', separately. That could either be due to the fact that either s or t is not in V (or in V'), or that the edges, in the separate graphs, simply do not *synchronize* to produce a path.

Case 1. We consider the case where s and t are in G but not in G'. A path is then created if, and only if:

$$\pi(I_G + \langle s, t\rangle) \cap \pi'(I_{G'}) \big/ \pi(I_G) \cap \pi'(I_{G'}) \neq 0$$

Checking the presence of such an inequality amounts to comparing the dimensions of $\pi(I_G + \langle s, t\rangle) \cap \pi'(I_{G'})$ and $\pi(I_G) \cap \pi'(I_{G'})$. Note that these are subspaces of $\mathbb{R}^{|C|}$, i.e. the computation is performed only over the common nodes. Furthermore, by fixing G and C, we can vary G' only computing $\pi'(I_{G'})$, without having to recompute additional information on G.

Case 2. We consider the case where s is in G but not in G' and t is in both. A path is then created if, and only if:

$$\pi(I_G + \langle s,t\rangle) \cap \pi'(I_{G'} + \langle t\rangle) \big/ \pi(I_G) \cap \pi'(I_{G'}) + \langle t\rangle \neq 0$$

Case 3. We consider the case where s is in G but not in G' and t is in G' but not in G. A path is then created if, and only if:

$$\pi(I_G + \langle s\rangle) \cap \pi'(I_{G'} + \langle t\rangle) \big/ \pi(I_G) \cap \pi'(I_{G'}) \neq 0$$

All other cases. In the general case, a path is created if, and only if:

$$\pi(I_G + J_G) \cap \pi'(I_{G'} + J_{G'}) \big/ \pi(I_G) \cap \pi'(I_{G'}) + J_{G^*} \neq 0$$

where:

$$J_G = \begin{cases} \langle s\rangle & \text{if } s \in V \text{ and } t \notin V \\ \langle t\rangle & \text{if } s \notin V \text{ and } t \in V \\ \langle s,t\rangle & \text{if } s \in V \text{ and } t \in V \\ 0 & \text{if } s \notin V \text{ and } t \notin V \end{cases}$$

$$J_{G'} = \begin{cases} \langle s\rangle & \text{if } s \in V' \text{ and } t \notin V' \\ \langle t\rangle & \text{if } s \notin V' \text{ and } t \in V' \\ \langle s,t\rangle & \text{if } s \in V' \text{ and } t \in V' \\ 0 & \text{if } s \notin V' \text{ and } t \notin V' \end{cases}$$

$$J_{G*} = \begin{cases} \langle s\rangle & \text{if } s \in C \text{ and } t \notin C \\ \langle t\rangle & \text{if } s \notin C \text{ and } t \in C \\ \langle s,t\rangle & \text{if } s \in C \text{ and } t \in C \\ 0 & \text{if } s \notin C \text{ and } t \notin C \end{cases}$$

We refer the reader to [Ada17g] for the details and some proofs on such characterizations. By defining bases, the dimensions may be computed using matrix operations. Those operations could also lend themselves to combinatorial interpretations given the nature of the problem. The direction will not be further pursued in this chapter.

2.8 Concluding remarks.

The chapter did not explicitely expound most of the mathematical connections that arise throughout. Its goal was to present the theory with the least amount of diversion possible. We end with three general remarks.

The maps between the systems, and their lifts, played an important role in the characterization. They did not, however, explicitely appear when interaction and generativity were initially defined. They were, nevertheless, always implicit in the partial order on the semilattice. The general level of the theory is then achieved by explicitely defining maps, or morphisms, between systems. In the general theory, those morphisms will be as important as (if not more important than) the systems themselves.

The lifts in this chapter have been described through vector spaces. Different linear (or *abelian*) objects ought to, however, be used to capture richer structures in the semilattice of systems and veils. The notion of exact sequences, and the results of the snake lemma, will remain in effects. Recovering information from the exact sequence will, however, not be as straightforward as it is in the case of vector spaces. It might require extra information from the systems. Our use of the snake lemma is, furthermore, only a special example of a mechanism relating the phenome of the combined system to the separate subsystems.

Finally, the tractabilty of the problem—coping with generative effects sustained by the veil—will depend on the tractability of the lifts. The same problem may possess different lifts. The *better* the lift is in capturing the structure of the problem, the *better* the linking solution is.

2.9 Appendix: On computability.

The ideas presented in this chapter, and the algebraic machinery, can be used to derive computable criteria for phenomes to emerge. As a certificate, we implemented a verbose program (in `GAP v.4.8.3`) of the path example. Below is a snapshot of the output of the program. The output is somewhat self-explanatory. The program takes two graphs G and H as input. The common nodes are then specified in both G and H. The endpoints A and E are also specified accordingly. The program then outputs whether or not there exists a path from A to E once the common nodes are identified. The answer is obtained via the characterization provided in the earlier section, on graphs living on different vertices.

```
-------------------------
 Info of G:
-------------------------
 Nb of nodes: 3
 Nb of edges: 1
 List of Edges: [ [ 1, 2 ] ]
 Nb of Components: 2
 Dim of H^1(G): 0

 No path in G.

-------------------------
 Info on H:
-------------------------
 Nb of nodes: 3
 Nb of edges: 1
 List of Edges: [ [ 2, 3 ] ]
 Nb of Components: 2
 Dim of H^1(H): 0

 No path in H.

-------------------------
 Info on Common:
-------------------------
 Nb of nodes: 3
 IDs in G: [ 1, 2, 3 ]
 IDs in H: [ 1, 2, 3 ]

-------------------------
 Distinguished nodes:
-------------------------
 A value of -1 indicates non-existence.
 A in G @: 1
 E in G @: 3
 A in H @: 1
 E in H @: 3

-------------------------
 Pulling Back
-------------------------
 Dim of pullback from G: 2
 Dim of pullback from H: 2
 Dim of Common System: 3
 Dim of H^1(Common): 1

-------------------------
 Decision Criterion:
-------------------------
 Dim of piH^1(G): 0
 Dim of piH^1(H): 0
 Dim of Kernel of F^3 --> piH^1(G)(+)piH^1(H): 3
 Quotienting subspace in H^1(Common): 2

 The program computes 4 objects:

 Dim of piI_G: 1
 Dim of piI_H: 1
 Dim of intersection: 0
 Augmented Dimension: 2    (1)
 Dim of pi(I_G+<A,E>): 3
 Dim of pi(I_H+<A,E>): 3
 Dim of intersection: 3    (2)

 Compare (1) and (2):  path emerges iff different.

#########################
 Path exists!
#########################

 Path emerged!
```

Modifying the graph H yields:

```
-------------------------
 Info of G:
-------------------------
 Nb of nodes: 3
 Nb of edges: 1
 List of Edges: [ [ 1, 2 ] ]
 Nb of Components: 2
 Dim of H^1(G): 0

 No path in G.

-------------------------
 Info on H:
-------------------------
 Nb of nodes: 3
 Nb of edges: 1
 List of Edges: [ [ 1, 2 ] ]
 Nb of Components: 2
 Dim of H^1(H): 0

 No path in H.

-------------------------
 Info on Common:
-------------------------
 Nb of nodes: 3
 IDs in G: [ 1, 2, 3 ]
 IDs in H: [ 1, 2, 3 ]

-------------------------
```

```
 Distinguished nodes:
-------------------------
 A value of -1 indicates non-existence.
 A in G @: 1
 E in G @: 3
 A in H @: 1
 E in H @: 3

-------------------------
 Pulling Back
-------------------------
 Dim of pullback from G: 2
 Dim of pullback from H: 2
 Dim of Common System: 2
 Dim of H^1(Common): 0

-------------------------
 Decision Criterion:
-------------------------
 Dim of piH^1(G): 0
 Dim of piH^1(H): 0
 Dim of Kernel of F^3 --> piH^1(G)(+)piH^1(H): 3
 Quotienting subspace in H^1(Common): 3

 The program computes 4 objects:

 Dim of piI_G: 1
 Dim of piI_H: 1
 Dim of intersection: 1
 Augmented Dimension: 3    (1)

 Dim of pi(I_G+<A,E>): 3
 Dim of pi(I_H+<A,E>): 3
 Dim of intersection: 3    (2)

 Compare (1) and (2):  path emerges iff different.

#########################
 No Path.
#########################
```

Finally, as an example of graphs defined on different sets of vertices, we get:

```
-------------------------
```

```
 Info of G:
-------------------------
 Nb of nodes: 13
 Nb of edges: 10
 List of Edges: [ [ 3, 5 ], [ 1, 7 ], [ 6, 7 ], [ 6, 12 ],
       [ 12, 2 ], [ 10, 8 ], [ 8, 9 ], [ 11, 9 ], [ 9, 4 ], [ 13, 4 ] ]
 Nb of Components: 3
 Dim of H^1(G): 2

 No path in G.

-------------------------
 Info on H:
-------------------------
 Nb of nodes: 17
 Nb of edges: 9
 List of Edges: [ [ 2, 4 ], [ 1, 5 ], [ 5, 13 ], [ 3, 7 ],
       [ 7, 9 ], [ 9, 15 ], [ 11, 9 ], [ 11, 13 ], [ 6, 10 ] ]
 Nb of Components: 8
 Dim of H^1(H): 7

 No path in H.

-------------------------
 Info on Common:
-------------------------
 Nb of nodes: 5
 IDs in G: [ 1, 2, 3, 4, 5 ]
 IDs in H: [ 1, 2, 3, 4, 5 ]

-------------------------
 Distinguished nodes:
-------------------------
 A value of -1 indicates non-existence.
 A in G @: 10
 E in G @: -1
 A in H @: -1
 E in H @: 9

-------------------------
 Pulling Back
-------------------------
 Dim of pullback from G: 3
 Dim of pullback from H: 2
 Dim of Common System: 4
 Dim of H^1(Common): 4

-------------------------
 Decision Criterion:
-------------------------
 Dim of piH^1(G): 2
 Dim of piH^1(H): 1
 Dim of Kernel of F^5 --> piH^1(G)(+)piH^1(H): 2
 Quotienting subspace in H^1(Common): 1

 The program computes 4 objects:

 Dim of piI_G: 2
 Dim of piI_H: 3
 Dim of intersection: 1
 Augmented Dimension: 1    (1)

 Dim of pi(I_G+<A,E>): 3
 Dim of pi(I_H+<A,E>): 4
 Dim of intersection: 2    (2)

 Compare (1) and (2):  path emerges iff different.

#########################
 Path exists!
#########################

 Path emerged!
```

Chapter 3

Where do cascades come from?

Abstract

We argue that the mathematical structure enabling cascade-like effects to intuitively emerge coincides with certain Galois connections. We introduce the notion of generative effects to formalize cascade-like phenomena. We define the notion of a veil, and show that such effects arise from either concealing mechanisms or forgetting characteristic from a system. We study properties of the veil, introduce dynamical veils and end by discussing factorizations and lifts. The goal is to initiate a mathematical base that enables us to further study such phenomena. In particular, generative effects will be linked to a certain loss of exactness. Homological algebra, and related algebraic methods, then come in to characterize such effects.

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3.1 Introduction.

We view cascade-like phenomena as not intrisic to the systems in play. They rather arise from a separation, between what we deem as observable from the system, and the counterpart of what is deemed to be concealed. Such phenomena will then appear once the behavior of the observable part cannot be explained by only what is seen, when a system is modified or made to interact with another. In such a setting, the parts of the system that are concealed, interact and produce more (or less) that what is deemed expected. The observable part of the system ought to be thought of as a property, a feature or a subpart of the system. Cascade-like phenomena then arise exactly when such a property does not behave well under modification of a system, or interconnection with another. Such instances are considered a nuisance in engineering practice, and our continual desire is to find modular, compositional means to interconnect and understand systems. It is however those nuisances that create the intuition of cascading behavior. They then should be a central component in a theory that aims to understand them.

3.1.1 A recipe for cascading phenomena.

One archetypical example of cascade effects emerges from a sequence of dominoes, minutiously set vertically one after the other. Topple the first in the sequence, and its sucessors come toppling down in an avalanche. But where does this intuition arise from? We argue that a situation of cascade-like phenomena requires two ingredients. The first ingredient is the presence and interplay of several systems: one monolithic system by itself cannot produce cascading phenomena. The second ingredient is a limited observation, on our part, of the system: cascading phenomena emerge only when we decide to not observe everything. To elucidate the two ingredients, let us consider the following two fictive situations. The aim of the fictive situations is to reveal *sensitive* issues when it comes to defining cascading phenomena, and potentially to reveal *anomalies* in our intuition regarding the effects. Dealing appropriately with each of the two issues are the ingredients, and they ought to be incorporated in a sound definition of cascade effects.

The first ingredient.

Let us suppose that our dominoes are really thin. We align the dominoes in a chain, each consecutive pair separated by a fixed distance d. If we topple the first domino, it falls and hits the second. The second falls in turn to hit the third. The falling sequence then goes on. Now suppose we start decreasing d. While the dominoes are separated, the intuition for cascading phenomena persists. But what happens when d becomes 0, where the dominoes are now touching? Are there any cascade effects?

If we suppose that cascade effects disappear, then why do they disappear? If the dominoes are moved apart by an imperceptible bit, the effects have to come back. Should the fact that they are meerly touching modify the presence of such a phenomenon? Alternatively, we can suppose that the effects remain. But then let us suppose that we glue the dominoes together, forming one thick domino. If we topple the *first* one, the rest come falling down. Is the intuition in this case still present?

The presence of the phenomenon hinges on having multiple systems interacting. In the case, where the dominoes are touching, whether separate or glued, thinking of them as a monolothic system forces the intuition to vanish. But if we explicitely think of them as separate parts (although glued) the intuition remains. A single domino falling by itself can then exhibit such a phenomena, if we split it (in our minds) to several pieces of thinner glued dominoes.

The second ingredient.

Let us conceive a mechanical device, equipped with a sequence of equally spaced slots. We vertically fix (or glue) in each slot one domino, making it immovable. The slots however are not rigid and can rotate, around a horizontal axis. Whenever the slot rotates, the domino it fixes rotates with it, again around a horizontal axis to yield a *fall* effect. The device is engineered so that, once it operates, the ith slot begins rotating at time $(i-1)T$ for some fixed time interval T. The interval T is fixed to be the time needed for the first domino to touch the second domino, once the first slot starts to rotate. As the slots are equally spaced, T denotes the time needed for an arbitrary domino to barely touch its successor. We now operate the device, and the first slot begins to rotate. The second slot begins to rotate after T. The ith slot begins to rotate after $(i-1)T$. The whole scheme consists of a *timed* mechanical device that successively rotates the slots, rotating the dominoes along the way. We can hardly claim any cascading effects while seeing and understanding the whole machinery in work.

However, let us suppose that we instead cover the mechanical device with a veil. The slots and machinery are fully concealed, and we can only observe the dominoes sticking out. Whatever is concealed under the veil does not exist for us anymore. Ideally, we cannot even know that something (if anything) is hidden. If the device (which we cannot know of its existence) begins to operate, the only thing we can ever observe is a sequence of dominoes toppling one another. Cascade effects would have emerged through the use of the veil. We may claim that the mechanical device is not a natural process. However, the physical laws of mechanics can be regarded themselves as the device. We thus recover the cascading intuition, by only focusing on the dominoes, and forgetting the physical laws. It is for such a reason why cascading phenomena are seen to arise in the case of the first ingredient where the dominoes are separated. We *non-intentionally* hide parts of the physical laws governing the whole setup when observing the dominoes.

Cascade effects cannot be conjured without the interplay of multiple systems. However, the interaction, or interconnection, of several systems only yields us an interconnected system. An interconnection theory by itself cannot explain interaction-related effects. Such effects only emerge when we conceal parts, or mechanisms, of the systems. In case we observe the whole systems, the intuition disappears. In case we observe nothing at all from the systems, the intuition also disappears. The intuition emerges whenever the observable parts of the separate systems fail to explain the observable part of the interconnected system.

3.1.2 Why do we ask such a question?

Our reason for asking such a question is not to indulge in philosphical musing. Our goal is to analyze cascade-like phenomena. The idea of a veil gives rise to a mathematical picture later on described. In the full generality of the theory, cascade-like effects can be linked to a certain loss of *exactness*. Homological algebra then comes into the setting to analyze the situation. Specifically, we can extract algebraic object from the systems that encode their potential to generate effects. Those objects can then be used to understand the phenomenon, and link the behavior of the interconnected system to its separate constituents. Mathematically, this picture enables us to develop cohomology theories to understand cascade-like effects. The 0th order cohomology objects encode the phenome of the system, and higher order objects encode potentials to produce effects.

3.1.3 The toy mathematical picture.

We will first substitute the term cascade effects with the more general term of interactional effects, namely effects that arise from the interaction of systems. Those can be informally understood as things that occur once systems interact, and that would not have occured without interaction. Cascading phenomena can then be loosely seen as a subclass of interactional effects.

To formally define interactional effects, we said we need two ingredients. First, we need a notion of interaction or interconnection of systems. The terms interaction and interconnection will be used synonymously. However, such a notion by itself cannot give rise to interactional effects. The interconnection of two systems only gives an interconnected system, and nothing more. The second ingredient then consists of equipping the theory of interconnection with a notion of interactional-effects. Such effects emerges when we single out what is observable from a system.

For simplicity, let S be a set and let $\oplus$ be a binary operator on S. The space $(S, \oplus)$ will be space of systems, where $s_1 \oplus s_2$ denotes the interaction of s_1 and s_2. The notion of interaction is then settled. An observation is then a map Φ from S to a set P. The set P ought to be thought of a space of simplified systems, and thus gains a notion of interaction.

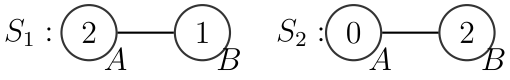
Interactional effects are then said to emerge whenever:

$$\Phi(s_1 \oplus s_2) \neq \Phi(s_1) + \Phi(s_2)$$

In this chapter, we will be concerned with a specific structure. We formalize the situation to one that admits a *generative* nature. The informal idea of *generativity* can be illustrated through the following example.

3.1.4 A contagion phenomenon.

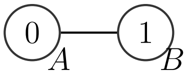
To dilute the abstraction and fill in mathematical details, we consider a situation of contagion. A system will consist of an undirected graph. Each node in the graph can be either infected (active or failed) or healthy (inactive or non-failed), and is assigned an integer k as a threshold. All nodes are initially healthy. A node then becomes infected if at least k of its neighbors are infected. Once a node is infected, it remains infected forever. In this case, the order of infections does not affect the final set of infected nodes. The system we described works on arbitrary undirected graphs, but for exposition, we will consider only systems defined on two nodes, as concrete examples. Specifically, let us consider the systems, S_1 and S_2:



The system S_1 can be summarized as “`if A is infected, then B becomes infected`”, while the system S_2 can be summarized as “node A is infected”. One can see that when the systems S_1 and S_2 are made to interact with each other, they will intuitively result in a cascade-like situation where B becomes infected. The question is: how do we formalize such an intuition? To formalize it we need two ingredients: a notion of interaction and a notion of *effects* equipped on top of interaction.

Having two systems interact, or equivalently interconnecting two systems, consists of keeping the minimum of the thresholds. The systems S_1 and S_2 interact to yield the system $S_1 \vee S_2$ given by:

$S_1 \vee S_2$:



Interaction can be understood as combining the description of the two systems. The system $S_1 \vee S_2$ is in fact summarized as:

“`if A is infected, then B becomes infected`
`AND`
`node A is infected`”.

The systems merge their update rules. The operator $\vee$ then gives us a notion of interaction through the merging of rules. This notion by itself however does not account for the cascading phenomenon.

To retrieve the intuition, we need to focus on a particular feature of our systems. Let $\Phi(S)$ denote the final set of infected nodes, that arise from system S. By focusing on the final set of infected nodes, we have discarded any dynamics in the system that could potentially lead to more infections. A set of infected nodes can be interpreted as a simplified system where thresholds are either 0 or ∞ depending on whether they are infected (0) or not (∞). They thus inherit a notion of interaction, of merging descriptions, which coincides with set union. Cascade-like intuition then arises because:

$$\Phi(S_1 \vee S_2) \neq \Phi(S_1) \cup \Phi(S_2)$$

Indeed, $\Phi(S_1) \cup \Phi(S_2) = \{A\}$ as $\Phi(S_1) = \{\}$ and $\Phi(S_2) = \{A\}$. However, $\Phi(S_1 \vee S_2) = \{A, B\}$. The observable part of the separate systems (i.e., their final set of infected nodes) fails to explain the observable part of the combined system. The discarded mechanisms interact in the full systems to produce new observable that cannot be accounted for.

More generally, let Σ denotes the set of nodes in the graph. If 2^Σ denotes the set of subsets of Σ, then the example lends itself to the following picture:

$$\left\langle 2^\Sigma, \cup \right\rangle \xleftarrow{\Phi} \left\langle \mathbf{System}, \vee \right\rangle$$

The effects are now encoded in the *inexactness* of the map Φ. The map Φ is not unstructured, and possesses certain properties. The chapter sets forth a thesis that cascade-like effects arise from situations akin to the diagrammatic representation above. The spaces of systems and phenomes vary, and are preordered sets. The map Φ tends to admit an adjoint, thus forming a Galois connection between systems and phenomes. Elements of the spaces interact to yield their join (i.e., their least upper-bound) whenever it exists. The effects are then sustained whenever Φ fails to commute with joins, i.e., the interaction operator.

3.1.5 Summary.

Generative effects are seen to arise from the pattern:

$$\left\langle \mathbf{Phenome}, \vee \right\rangle \xleftarrow{\Phi} \left\langle \mathbf{System}, \vee \right\rangle.$$

The space $\langle \mathbf{System}, \vee \rangle$ (resp. $\langle \mathbf{Phenome}, \vee \rangle$) is a preordered set of systems (resp. of phenomes) where every pair of systems (resp. of phenomes) interact to yield their least upper-bound, via $\vee$, the join operator. The map Φ acts as the described veil, partially concealing the systems and leaving the phenome bare. It admits an adjoint, thus forming a Galois connection. Its adjoint recovers from a phenome the simplest system explaining it. Generative effects are said to be sustained whenever:

$$\Phi(S \vee S') \neq \Phi(S) \vee \Phi(S').$$

They are sustained whenever the phenome of the combined system cannot be explained by the phenomes of the separate systems.

3.1.6 Outline of the chapter

The chapter expounds this formalization. It justifies the choices made, and equips the mathematics with the needed intuition. We continue to elucidate the example on contagion in Section 3. We introduce the notion of a veil and generative effects in Section 4. Those can be seen to emerge from either concealing mechanisms in the systems (developed in Section 5) or forgetting characteristics (developed in Section 6). Indeed, we establish in Section 7 that every veil can be factored into an instance of these two cases, and discuss its relation to Galois connections. We introduce, in Section 8, the notion of a dynamical veil to capture temporal aspects in cascade effects. We finally develop techniques of factorization and lifts, in Section 9, to retrieve veils from *non-veils*, and end with some remarks in Section 10.

3.1.7 The goal of this line of research

A fuller, more englobing, development of the concepts can be performed via the use of categories. We however restrict to preordered sets to not introduce unnecessary complications for the readers. Preorders can be trivially regarded as categories, and are thus a special case of the general concept. Nevertheless, most of the analysis provided in this chapter can be extended out to the more general case. For more details on the general case, we refer the reader to [Ada17g]. In the full generality of the theory, generative effects are linked to a certain *loss of exactness*. Homological algebra then comes into the setting to analyze the situation. Specifically, we can extract algebraic object from the systems that encode their potential to generate effects. Those objects can then be used to understand the phenomenon, and link the behavior of the interconnected system to its separate constituents. Mathematically, this picture enables us to develop (co)homology theories to understand cascade effects. The 0th order (co)homology objects encode the phenome of the system, and higher order objects encode potentials to produce effects. We refer the reader to [Ada17i] for a thorough development of this line of research.

3.2 Mathematical preliminaries and definitions.

A preordered set or *proset* $\langle S, \leq \rangle$ is a set S equipped with a (binary) relation $\leq$ that is reflexive and transitive. If $\leq$ is also antisymmetric, then $\leq$ becomes a partial order and $\langle S, \leq \rangle$ becomes a partially ordered set or *poset*. A proset is said to be a join-semilattice (resp. meet-semilattice) if every pair of elements admits a least upper-bound, termed *join* (resp. greatest lower-bound, termed *meet*). A proset that is both a join-semilattice and a meet-semilattice is said to be a lattice. Note that if a proset $\langle S, \leq \rangle$ admits finite joins (resp. finite meets) then $\leq$ is antisymmetric (see e.g., Proposition 3.4.2 for more details).

A proset is said to be finitely cocomplete (resp. finitely complete) if every finite subset of it admits a join (resp. meet). A finitely cocomplete (resp. finitely complete) proset is then only a join-semilattice (resp. meet-semilattice) with a minimum (resp. maximum) element. A proset S is said to be cocomplete (resp. complete) if every subset of it admits a join (resp. meet). A cocomplete proset is then necessarily complete: the lower-bounds of a subset admit a join by cocompleteness. The converse also holds. A complete lattice is then a lattice that admits arbitrary meets and joins. A complete lattice is thus equivalently a cocomplete (resp. complete) preordered set.

3.2.1 Notation.

If S is a set, then 2^S denotes the set of subsets of S. If S and T are sets (resp. preordered sets), then S^T denotes the set of maps (resp. order-preserving maps) from T to S. If S and T are prosets, then the set S^T inherits a natural preorder relation $f \leq g$ if, and only if, $f(t) \leq g(t)$ for all $t \in T$.

3.3 The contagion phenomenon, revisited.

The example presented in the introduction is only an instance of a more general class of systems. A system in concern consists of n nodes, or parts. Each node can be either infected (active or failed) or healthy (inactive or non-failed), and is attributed a collection of neighborhood sets. A neighborhood set is only a subset (possibly empty) of the n nodes. Each node can be attributed either one, multiple or no neighborhood sets. A node becomes infected if all the nodes in (at least) one of its neighborhood sets are infected. Once a node becomes infected, it remains infected forever. Again, the order of infections does not affect the set of final infected nodes.

The example presented in the introduction can be seen as a special case where the neighborhood sets of node i are only subsets of cardinality k, the threshold of i. The operator Φ described, and its *inexactness*, carries through unchanged.

3.3.1 Syntax and interpretation.

Let $\Sigma := \{a, b, \cdots, h\}$ be a finite set of n elements. The set 2^Σ denotes the set of subsets (or powerset) of Σ. For notational convenience, we define **D** (for Description) to be the set:

$$\Sigma \to 2^{2^{\Sigma}}$$

A map in $\mathbf{D}$ assigns to every element of Σ a collection of subsets of Σ. A system, as presented at the start of the section, is syntactically described by a map $\mathcal{N} \in \mathbf{D}$. Conversely every map in $\mathbf{D}$ is a meaningful syntactic description of a system.

Interpretation.

The syntactical description $\mathcal{N}$ is interpreted as follows. A system is made up of n part, labeled say $a, b, \cdots, h$. To each part i is assigned a collection of neighborhood sets $\mathcal{N}(i)$. Every part can be either infected (active or failed) or healthy (inactive or non-failed). All parts are initially inactive. Part i is infected at time $m+1$ if, and only if, either it was infected at time m or all the parts in some neighborhood set $N \in \mathcal{N}(i)$ of i are infected at time m. Thus, once a part is infected, it remains infected forever.

Let A_m denote the set of infected parts at time m. We initiate A_0 to be the empty set, and recursively define $A_1, A_2, \cdots$ such that $i \in A_{m+1}$ if, and only if, either $i \in A_m$ or $N \subseteq A_m$ for some set $N \in \mathcal{N}(i)$. Therefore, every map in $\mathbf{D}$ assigns to part i a monotone (or order-preserving) Boolean function $\phi_i : 2^{\Sigma} \to 2^{\{*\}}$. The set $\{*\}$ denotes the set with one element. Then:

$$i \in A_{m+1} \quad \text{iff} \quad \text{either } i \in A_m \text{ or } \phi_i(A_m) = \{*\}.$$

Whenever the set Σ is finite, the dynamics converge after finitely many steps.

Proposition 3.3.1. *If Σ has cardinality n, then $A_n = A_{n+1}$.*

Proof. If $A_n \neq A_{n+1}$, then $A_m \neq A_{m+1}$ for $0 \leq m \leq n-1$. Thus, if $A_n \neq A_{n+1}$ then A_n would contain more than n elements. □

We thus refer to the final set of infected nodes as A_∞. The set A_∞ is only used to correspond to the case where A_0 is initialized to the empty set.

Interaction.

Two systems are made to syntactically interact by merging their descriptions. Syntactic descriptions $\mathcal{N} \in \mathbf{D}$ and $\mathcal{N}' \in \mathbf{D}$ interact by yielding their union $\mathcal{N} \cup \mathcal{N}' \in \mathbf{D}$ where $(\mathcal{N} \cup \mathcal{N}')(i) = \mathcal{N}(i) \cup \mathcal{N}'(i)$. The collection of neighborhoods are combined. Indeed, we can order $\mathbf{D}$ by inclusion as $\mathcal{N} \subseteq \mathcal{N}'$ if $\mathcal{N}(i) \subseteq \mathcal{N}'(i)$ for all i. Every pair of descriptions $\mathcal{N}$ and $\mathcal{N}'$ in the partially ordered set $\mathbf{D}$ admits a least upper-bound denoted by $\mathcal{N} \cup \mathcal{N}'$.

3.3.2 Semantics.

To study the systems, we will recover from a syntactical description $\mathcal{N} \in \mathbf{D}$, a map $f_{\mathcal{N}} : 2^{\Sigma} \to 2^{\Sigma}$ that sends $S \subseteq \Sigma$ to the final set of infected nodes if A_0 was initialized to S, i.e., if all the parts in S are initially infected.

Proposition 3.3.2. *The set A_∞ corresponding to the final set of infected nodes is $f_{\mathcal{N}}(\emptyset)$.*

Proof. The proof is immediate by definition of A_∞. □

As mentioned, the set A_∞ is only used to correspond to the case where A_0 is initialized to the empty set.

The map $f_{\mathcal{N}}$ derived from a syntactical description $\mathcal{N}$ may be thought of as being the object that gives a meaning to the description, the semantics behind the syntax. Different syntactical descriptions in $\mathbf{D}$ may yield the same system object. However, different system objects refer to different systems, when it comes to looking at the final set of infected nodes. The system object $f_{\mathcal{N}}$ can be seen as the representation-independent object we are interested in. We will thus refer, in this section, to $f_{\mathcal{N}}$ as the system (object), as opposed to $\mathcal{N}$ which is referred to as the system syntax or description.

Definition 3.3.3. *A map $f : 2^\Sigma \to 2^\Sigma$ is said to satisfy A.1, A.2 and A.3, respectively, if:*

A.1. $S \subseteq f(S)$ *for all* $S \subseteq \Sigma$.

A.2. *If* $S \subseteq S'$ *then* $f(S) \subseteq f(S')$, *for all* $S, S' \subseteq \Sigma$.

A.3. $ff(S) = f(S)$ *for all* $S \subseteq \Sigma$.

Proposition 3.3.4. *If $\mathcal{N} \in \mathbf{D}$, then $f_{\mathcal{N}} : 2^\Sigma \to 2^\Sigma$ satisfies A.1, A.2 and A.3.*

Proof. The axioms A.1, A.2 and A.3 immediately follow from the description of the systems. We refer the reader to [Ada17b] for more details. □

Conversely, we have:

Proposition 3.3.5. *If $f : 2^\Sigma \to 2^\Sigma$ satisfies A.1, A.2 and A.3, then $f = f_{\mathcal{N}}$ for some description $\mathcal{N} \in \mathbf{D}$.*

Proof. Construct $\mathcal{N}$ such that $\mathcal{N}(i) = \big\{S \in 2^\Sigma : i \in f(S)\big\}$. □

We then define **System** to be the set of maps satisfying A.1, A.2 and A.3. The maps in **System** are often known as *closure operators*. On one end, they appeared in the work of Tarski (see e.g., [Tar36] and [Tar56]) to formalize the notion of deduction. On another end, they appeared in the work of Birkhoff, Ore and Ward (see e.g., [Bir36], [Ore43] and [War42], respectively), parts of foundational work in universal algebra. The first origin reflects the consequential relation in the effects considered. The second origin reflects the theory of interaction of multiple systems. Closure operators appear as early as [Moo10].

If we order **System** by:

$$f \leq g \quad \text{if} \quad f(S) \leq g(S) \text{ for all } S \subseteq \Sigma,$$

then $\langle \textbf{System}, \leq \rangle$ becomes a partially ordered set. The relation $f \leq g$ can be thought of as f is a subsystem of g. Furthermore, every pair of systems f and g admits an upper-bound $f \vee g$.

Proposition 3.3.6. *If Σ has cardinality n, then $f \vee g = (fg)^n$.*

Proof. The map $(fg)^n$ satisfies A.1, A.2 and A.3, and belongs to **System**. Indeed, A.1 and A.2 are preserved by composition. The axiom A.3 is satisfied as $(fg)^n(S) = (fg)^{n+1}(S)$ whenever Σ has cardinality n, by an argument similar to that in the proof of Proposition 3.3.1. Finally, if $h \in \mathbf{System}$ and $f \vee g \leq h$, then $hf = fh = h$ and $hg = gh = h$. Thus $(fg)^n \leq (fg)^n h \leq h$ whenever $f \vee g \leq h$. □

The poset **System** is then a join-semilattice $\langle \mathbf{System}, \leq, \vee \rangle$. The semilattice also admits meets (i.e., greatest lower-bounds) making it a lattice. Its minimum element is the identity map, while its maximum element is the map $- \mapsto \Sigma$.

Two systems can be seen to interact (semantically) by iteratively applying their system maps till they yield an idempotent map, i.e., satisfying A.3. The properties A.1 and A.2 are always preserved under composition. Most importantly, the semantical interaction of systems coincides with the syntactical interaction of systems.

Proposition 3.3.7. *If $\mathcal{N}$ and $\mathcal{N}'$ are descriptions, then $f_{\mathcal{N} \cup \mathcal{N}'} = f_{\mathcal{N}} \vee f_{\mathcal{N}'}$.*

Proof. We have $f_{\mathcal{N} \cup \mathcal{N}'}(S) = S$ if, and only if, whenever $N \in \mathcal{N} \cup \mathcal{N}'(i)$ lies in S, then $i \in S$. Or equivalently if, and only if, $f_{\mathcal{N}}(S) = f_{\mathcal{N}'}(S) = S$. Furthermore, the fixed-points of $f_{\mathcal{N}} \vee f_{\mathcal{N}'}$ are the sets that are fixed-points of both $f_{\mathcal{N}}$ and $f_{\mathcal{N}'}$. The result then follows as the maps in **System** are uniquely determined by their fixed-points. See e.g., Theorem 3.5.5 or [Ada17c] for more details on the last assertion. □

We established thus far a theory of interconnection, via the space $\langle \mathbf{System}, \vee \rangle$. However, interconnecting two systems will only give us an interconnected system. No cascading phenomena are yet present. Those will only emerge once we decide to conceal features in the systems.

3.3.3 The contagion intuition.

To recover the intuition, we conceal the dynamics. We do so by only keeping the final set of infected nodes. We are thus observing from our systems, subsets of Σ corresponding to the final set of infected nodes. To this end, we define $\Phi : \mathbf{System} \to 2^{\Sigma}$ to be the map sending f to its least fixed-point. Such a map is well defined, as:

Proposition 3.3.8. *If $f : 2^{\Sigma} \to 2^{\Sigma}$ satisfies A.1, A.2 and A.3, then its set of fixed-points $\mathrm{fix}(f) = \{S : fS = S\}$ when ordered by inclusion forms a complete lattice. Furthermore, if S and S' are fixed-points of f, then $S \cap S'$ is a fixed-point of f.*

Proof. Let $S, S' \in \mathrm{fix}(f)$ be fixed-points. We have $f(S \cap S') \leq f(S) = S$ and $f(S \cap S') \leq f(S') = S'$ by A.2. As $S \cap S' \leq f(S \cap S')$ by A.1, we get $f(S \cap S') = S \cap S' \in \mathrm{fix}(f)$. Let $\mathcal{S}$ denote the collection of fixed-points that contain both S and S', namely:

$$\mathcal{S} := \{T \in \mathrm{fix}(f) : S \subseteq T \text{ and } S' \subseteq T\}$$

As $\Sigma \in \mathrm{fix}(f)$, the set $\mathcal{S}$ is non-empty. The least-upper-bound of S and S' in $\mathrm{fix}(f)$ is then $\bigcap \mathcal{S}$, the intersection of all the sets in $\mathcal{S}$. The greatest lower-bound of S and

S' is $S \cap S'$. The set $\text{fix}(f)$ then forms a lattice. The lattice $\text{fix}(f)$ is complete as it is finite. □

The map Φ is then well defined, as a complete lattice admits a minimum element. This minimum element corresponds to the set-intersection of all the fixed-points of f.

We term our observations, namely the subsets of Σ, as phenomes. We also refer to 2^{Σ} as the space of phenomes, denoted by **Phenome**.

Proposition 3.3.9. *The map* $\Phi : \mathbf{System} \to \mathbf{Phenome}$ *satisfies:*

P.1. If $f \leq g$ *in* **System**, *then* $\Phi(f) \subseteq \Phi(g)$.

P.2. If $S \in \mathbf{Phenome}$, *then* $\{f : S \subseteq \Phi(f)\}$ *has a (unique) minimum element.*

Proof. (P.1) If $f \leq g$ then $\{S : fS = S\} \supseteq \{S : gS = S\}$. (P.2) For every S, the system $- \mapsto - \cup S$ is the minimum element of $\{f : S \subseteq \Phi(f)\}$. □

First, the map Φ is order-preserving, and thus preserves the subsystems relation among the systems. Second, every set of infected nodes can be lifted to a simplest system explaining that set.

Such a map Φ, satisfying P.1 and P.2, from the space of systems to a space of phenomes is termed a *veil*. In this context, contagion phenomena (later termed generative effects) arise precisely whenever:

$$\Phi(f \vee g) \neq \Phi(f) \vee \Phi(g).$$

They arise whenever keeping only the final set of infected nodes cannot account for what happens when the two systems interact. Indeed, the mechanisms that we have concealed interact and activate, or infect, more nodes than we can observably account for. The phenomenon is now encoded in the *inexactness* of the veil Φ. The inequality is the essential point.

We return to this example as the chapter unfolds. We first set out to describe the general structure of the situation, and formally introduce generative effects.

3.4 The two ingredients, formalized.

As illustrated, two ingredients are required to sustain generative effects. We first need a theory of interaction or interconnection of systems. A theory of interconnection by itself cannot, however, account for such phenomena. We need a notion of a veil, that conceals features from a system, and keeps a phenome observable. Generative effects then emerge whenever the phenome of the combined system cannot be explained by the phenomes of the separate systems.

3.4.1 Interaction of systems.

Let **System** be a preordered set, namely a set equipped with a binary relation $\leq$ that is reflexive and transitive. Each element of **System** is considered to be a system. The $\leq$ relation dictates how the systems are related to each other.

Definition 3.4.1. *A system s is said to be a subsystem of s' if $s \leq s'$.*

Two systems will interact to yield their least upper-bound, only if it exists. We will generally consider that least upper-bounds of finite subsets always exist, as such conditions will (or can be made to) be satisfied in most of our situations in concern. A preordered set is said to be finitely cocomplete, if every finite subset of it admits a (unique) least upper-bound.

Proposition 3.4.2. *If* **System** *is finitely cocomplete, then the relation $\leq$ is antisymmetric.*

Proof. If $s \leq s'$ (resp. $s' \leq s$) then s' (resp. s) is the least upper-bound of s and s'. As the least upper-bound is unique by definition, we get $s = s'$ whenever $s \leq s'$ and $s' \leq s$. □

If **System** is finitely cocomplete, then $\leq$ becomes a partial order. **In this chapter, we consider System to be a finitely cocomplete preordered set, unless indicated otherwise.** A finitely cocomplete preordered set always admits, by definition, a minimum element: the least upper-bound of the empty set.

Definition 3.4.3. *Two systems s and s' interact to yield their least upper-bound $s \vee s'$. More generally, a finite subset of systems $S \subseteq$* **System** *interacts to yield its least upper-bound $\vee S$ as a resulting system.*

A collection of systems can only interact in a unique way and it is via the $\vee$ operator, to yield their least upper-bound. The binary operator $\vee$ is associative, commutative and idempotent. The algebra $\langle \textbf{System}, \leq, \vee \rangle$ is usually termed a join-semilattice.

Remark.

Conversely, every associative, commutative and idempotent binary operator on a set induces a partial order on it. The development could have thus began with a join semilattice. However, the order relation is seen to be more essential than the join operation. This is especially true in the general level of the developed theory, through the use of categories and functors. In the general case, the order relation is replaced by sets of morphisms and joins are replaced by colimits. This direction will however not be considered in this chapter.

Remark.

The subsystem relation may admit various interpretations. The notion of interconnection advocated by the behavioral approach to systems theory (see e.g., [PW98], [Wil07], and Subsection 3.5.3) can be seen as a special case of that developed in this section. Indeed, the notion of a subsystem in this section translates to a reverse inclusion of behaviors. Furthermore, interpreting $s \leq s'$ as s being a partial description or an approximation of s' is reminiscent of ideas in [Sco71], [Sco72] and [SS71]. The implication of such a connection will not however be investigated in this chapter. Such a direction of research may however well be fruitful.

3.4.2 Interlude on capturing the generativity of a system.

Let us informally consider a generative grammar (see e.g. [Cho65] and related work for a formal treatment) to be a collection of rules that dictates which sentences can be formed. Every grammar then builds one language, and different grammars may describe the same language. The grammars generating a same language are, however, different: adding a rule to one grammar could yield a very different effect on the language than adding it to another grammar. Similar effects occur in deduction (as seen through contagion in Section 3.3) and in situations exhibiting cascade effects or emergent phenomena. How do these grammars then gain this *generativity*? It is definitely coming from their grammar rules. Yet, how do we capture it? We capture it by *destroying* the rules, and studying how the grammar in full and the grammar without the rules (amounting to only the language) behave when combined with other grammars. It is the vivid discrepancy in interaction outcome between the presence of the rules and their absence that encodes the generativity. To then capture cascade effects resulting from the interaction of systems, we perform the following experiment. On one end, we let the systems interact and observe the outcome of the interaction. On another end, we destroy the potential a system has to produce effects let them interact without it. These two ends, in the presence of cascade effects, will show a discrepancy in interaction outcome. This discrepancy then encodes the phenomenon. Studying the discrepancy amounts to studying the phenomenon.

3.4.3 Veils and generative effects.

Generative effects are seen to emerge when we decide to focus on a particular property of a system. Such a focus is achieved by declaring a map $\Phi : \mathbf{System} \to P$, termed a *veil*, from the set of systems, to a set of observables, termed *phenomes*. Phenomes can be properties, features or even subsystems of a particular system. They ought to be thought of as simplified systems, and thus inherit an order-relation and a notion of interaction. The space P is then, in turn, a cocomplete preordered set.

Definition 3.4.4. *A veil on* $\mathbf{System}$ *is a pair* (P, Φ) *where* $\langle P, \leq \rangle$ *is a finitely cocomplete preordered set, and* $\Phi : \mathbf{System} \to P$ *is a map such that:*

V.1. The map Φ *is order-preserving, i.e., if* $s \leq s'$*, then* $\Phi s \leq \Phi s'$*.*

V.2. Every phenome admits a simplest system that explains it, i.e., the set $\{s : p \leq \Phi s\}$ *has a (unique) minimum element for every* $p \in P$.

As the map Φ always subsumes a codomain, we will often refer to Φ as the veil, instead of the pair (P, Φ). However, viewing a veil as a pair (P, Φ) highlights an important point. We may define different veils for a same space of systems, and each veil would define the space of phenomes to be observed from the system. The picture to keep in mind then is not that of fixing **System** and P and varying a veil in between. It is of fixing **System** and varying (P, Φ) to yield different facets of the systems.

The veil is intended to hide away parts of the system, and leave other parts, the phenome, of the system bare and observable. The axiom V.1 indicates that veiling a subsystem of a system may only yield a subphenome of the phenome of the system. The axiom V.2 indicates that everything one observes can be completed in a simplest way to something that extends under the veil. Generative effects occur precisely when one fails to explain the happenings through the observable part of the system. In those settings, the things concealed under the veil would have interacted and produced observable phenomes.

Definition 3.4.5. *A veil* (P, Φ) *is said to sustain generative effects if* $\Phi(s \vee s') \neq \Phi(s) \vee \Phi(s')$ *for some* s *and* s'.

Different veils may be defined for the same space **System**. Some will sustain generative effects and some will not. For instance, both veils $(\textbf{System}, \mathrm{id})$ and $(\{*\}, * : \textbf{System} \to \{*\})$ do not sustain generative effects at all. The veil $(\textbf{System}, \mathrm{id})$, being the identity map, hides nothing, while the veil $(\{*\}, * : \textbf{System} \to \{*\})$ hides everything. All that can be observed is explained by what is already observed. Thus the standard intuition for systems exhibiting cascading phenomena, or contagion effects, does not stem from a property of a system. It is rather the case that the situation admits a highly suggestive phenome and a highly suggestive veil that sustains such effects. Those effects are thus properties of the situation. Should we change the veil, we may either increase those effects, diminish them or even make them completely go away. Such interaction-related effects depend only on what we wish to observe.

The first property of the veil is somewhat self-explanatory. It ensures that the map respects the relation among systems and is compatible with the preorders. The second property, is less transparent, but gives the map a *generative* intuition present in cascading phenomena. To explain the second property, we note that generative effects can be seen to arise from two situations. We either conceal mechanisms in the systems, or we forget characteristics of the systems. These two situations will be expounded in the next two sections.

To make the space of phenomes P explicit in the chapter, as done with **System**, we will often refer to P as **Phenome**. Such a reference is mainly done in the following two sections. **Similarly to System, we consider Phenome in this chapter to be a finitely cocomplete preordered set, unless indicated otherwise.**

3.5 Concealing mechanisms.

The first source of generative effects consists of concealing mechanisms, or dynamics, in a given system. Two systems, sharing a same phenome, may become identical once the mechanisms are concealed. Their potential to produce effects in the phenome, while interacting with other systems, may however be different. Indeed, concealed mechanisms may play a role upon the interaction of systems.

To conceal (or destroy) mechanisms in a system, we require a map $\kappa : \mathbf{System} \to \mathbf{System}$ satisfying:

K.1. $\kappa(s) \leq s$ for all s.

K.2. If $s \leq s'$, then $\kappa(s) \leq \kappa(s')$, for all s and s'.

K.3. $\kappa\kappa(s) = \kappa(s)$ for all s.

First, the map κ reduces a system to a subsystem of it. Second, the map κ preserves the relation among systems. Third, the map κ does not discard anything from a system whose mechanisms are already discarded. An operator satisfying K.1, K.2 and K.3 are usually termed kernel operators.

The operator κ can be intuitively expected to sustain generative effects whenever $\kappa(s \vee s') \neq \kappa(s) \vee \kappa(s')$ for some s and s'. In such a case, the mechanisms concealed interact and produce more than what can only be produced by the phenomes. Put differently, the discarded mechanisms have a role to play in the interaction of systems with respect to our simplistic view, as phenomes, of the systems.

Let us define $\mathbf{Phenome} \subseteq \mathbf{System}$ to be the set of fixed-points $\{s : \kappa(s) = s\}$ of κ. Then by K.3, we get $\kappa(\mathbf{System}) = \mathbf{Phenome}$. We may then define $\pi : \mathbf{System} \to \mathbf{Phenome}$ such that $\pi(s) = \kappa(s)$. Every kernel operator on **System** thus gives rise to a surjective veil:

Proposition 3.5.1. *The map π is surjective and order-preserving, and for every $p \in \mathbf{Phenome}$, the set $\{s : p \leq \pi(s)\}$ has a minimum element.*

Proof. As $\pi(p) = \kappa(p) = p$ for every $p \in \mathbf{Phenome}$, the map π is surjective. The map π is clearly order-preserving. Finally, the set $\{s : p \leq \pi(s)\}$ has p itself as a minimal element. □

Conversely, every surjective veil induces a kernel operator on **System**:

Proposition 3.5.2. *If $\pi : \mathbf{System} \to \mathbf{Phenome}$ is a surjective order-preserving map such that $\{s : p \leq \pi(s)\}$ has a minimum element for every p, then there exists a unique injective order-preserving map $i : \mathbf{Phenome} \to \mathbf{System}$ such that πi is the identity map on **Phenome**, and $i\pi$ is a kernel operator on **System**.*

Proof. For every $p \in \mathbf{Phenome}$ define $i(p)$ to be the minimum element of $\{s : p \leq \pi(s)\}$. As π is surjective, it follows that πi is the identity. The map i is then injective. The map $i\pi$ is a kernel operator as $i\pi(s) = \min\{s' : \pi(s) \leq \pi(s')\}$. The requirements K.i can then be easily checked. Uniqueness of i follows from Proposition 3.7.4(i). □

Finally, whether or not generative effects are sustained by the veil π, depends on the properties of the kernel operator κ.

Proposition 3.5.3. *If $s, s' \in$ **System**, then:*

$$\pi(s \vee_{\mathbf{System}} s') \neq \pi(s) \vee_{\mathbf{Phenome}} \pi(s') \textit{ iff } \kappa(s \vee_{\mathbf{System}} s') \neq \kappa(s) \vee_{\mathbf{System}} \kappa(s').$$

Proof. If s and s' are fixed-points of κ, then their join in **System** coincides with their join in **Phenome**. Indeed, if $\kappa(s) = s$ and $\kappa(s') = s'$, then $\kappa(s \vee s') = s \vee s'$ by K.1 and K.2. □

We next provide some example situations of concealing mechanisms.

3.5.1 Contagion and deduction systems.

We return to our contagion example. Recall that a system corresponds to a map $f : 2^\Sigma \to 2^\Sigma$ satisfying:

A.1. $S \subseteq f(S)$ for all S.

A.2. $f(S) \subseteq f(S')$ if $S \subseteq S'$ for all S and S'.

A.3. $ff(S) = f(S)$ for all S.

The poset **System** corresponds to the set of maps satisfying A.1, A.2 and A.3, ordered by $f \leq g$ if $f(S) \leq g(S)$ for all S. Every pair $f, g \in$ **System** admits a least upper-bound $f \vee g \in$ **System**. The poset **Phenome** corresponds to 2^Σ ordered by inclusion. Two sets in 2^Σ admit their set-union as the least upper-bound. We then define $\pi :$ **System** $\to$ **Phenome** that sends f to its least fixed-point.

Proposition 3.5.4. *The map π is a surjective veil.*

Proof. The map π is surjective as it maps every system $- \mapsto - \cup S$ into S. The set $\{s : S \leq \pi(s)\}$ also has $- \mapsto - \cup S$ as a minimum element. It is also clearly order-preserving. □

If we define $i :$ **Phenome** $\to$ **System** to be the map $S \mapsto - \cup S$, then $i\pi$ yields a kernel operator. This kernel operator can be interpreted as destroying all the potential a system has to infect additional nodes when interacting with other systems. The kernel operator yields the simplest system (with respect to the dynamics) that can account for the infected nodes at the end.

A dual perspective.

Let $\mathcal{L}$ denote the collection of subsets of 2^Σ such that (i) $2^\Sigma \in \mathcal{L}$ and (ii) if $A, B \in \mathcal{L}$ then $A \cap B \in \mathcal{L}$. The sets in $\mathcal{L}$ are sometimes called *Moore families* or *closure systems.* The set $\mathcal{L}$ may then be ordered by reverse inclusion, to yield a lattice whose join is set-intersection.

Theorem 3.5.5. *The map $f \mapsto \{S : f(S) = S\}$ defines an isomorphism between $\langle \mathbf{System}, \leq, \vee \rangle$ and $\langle \mathcal{L}, \supseteq, \cap \rangle$*

Proof. Such a fact is well known regarding closure operators. We refer the reader to [Ada17c] for the details. □

As such, every system can be uniquely identified with its set of fixed-points. Interaction of systems then consists of intersecting the fixed-points. As a consequence:

Corollary 3.5.6. *$\pi(f)$ is the intersection of all the fixed-points of f.* □

For more details on the properties of such systems, we refer the reader to [Ada17c]. This line of example first aimed at understanding the mathematical structure underlying models of diffusion of behavior commonly studied in the social sciences. The setup there consists of a population of interacting agents. In a societal setting, the agents may refer to individuals. The interaction of the agents affect their behaviors or opinions. The goal is to understand the spread of a certain behavior among agents given certain interaction patterns. Threshold models of behaviors (captured by M.0, M.1, M.2 and M.3 in [Ada17c]) have appeared in the work of Granovetter [Gra78], and more recently in [Mor00]. Such models are key models in the literature, and have been later considered by computer scientists, see. e.g., [Kle07] for an overview.

3.5.2 Relations and projections.

A relation R between sets A and B is a subset of $A \times B$. The set of relations between A and B, ordered appropriately, admits two canonical veils.

First, define $\pi : (2^{A \times B}, \subseteq) \to (2^A, \subseteq)$ to send R to $\{a : (a, b) \in R \text{ for all } b\}$. The map π is a surjective veil. Indeed, the set $\{R : S \subseteq \pi R\}$ has $S \times B$ as a minimum element. This veil sustains generative effects as generally:

$$\pi(R \cup R') \neq \pi(R) \cup \pi(R')$$

Second, define $\pi' : (2^{A \times B}, \supseteq) \to (2^A, \supseteq)$ that sends R to $\{a : (a, b) \in R \text{ for some } b\}$. The map π' is also a surjective veil. Indeed, the set $\{R : S \supseteq \pi' R\}$ has $S \times B$ as a *minimum* element with respect to $\supseteq$. This veil also sustains generative effects as generally:

$$\pi(R \cap R') \neq \pi(R) \cap \pi(R')$$

The posets $(2^{A \times B}, \subseteq)$ and $(2^{A \times B}, \supseteq)$ of systems are dual to each other. The join corresponds to set-union in the first, whereas it corresponds to set-intersection in the second. The veil π' can be further interpreted in systems-theoretic situations, through the next example.

3.5.3 Subsystem behavior or concealing parameters.

In the behavioral approach to systems theory, a system is viewed as a pair of sets $(\mathbb{U}, \mathcal{B})$. The set $\mathbb{U}$—termed, the *universum*—depicts the set of all possible outcomes or trajectories. The set $\mathcal{B}$ is a subset of $\mathbb{U}$—termed the *behavior*—that defines which outcomes are deemed allowable by the dynamics of the system. The sets $\mathbb{U}$ and $\mathcal{B}$ can be equipped with various mathematical structures to suit various need. We will however, without loss of generality, only be concerned with sets, without any additional structure. Interconnecting two systems $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ yields the system $(\mathbb{U}, \mathcal{B} \cap \mathcal{B}')$. Indeed, the interconnected systems keeps the trajectories that are deemed possible by the separate systems. We refer the reader to [PW98] and [Wil07] for more details on the behavioral approach.

In this subsection, we are interested in the behavior of a subsystem of a system $(\mathbb{U}, \mathcal{B})$ as some changes are incurred into the greater system. A change in this setup is depicted as another system $(\mathbb{U}, \mathcal{C})$. Incurring the change then consists of obtaining the system $(\mathbb{U}, \mathcal{B} \cap \mathcal{C})$. To define a subsystem, we project $\mathbb{U}$ onto a smaller universum. For instance, let us suppose $\mathbb{U} = \mathbb{S} \times \mathbb{S}'$ is a product space. Projecting $\mathcal{B}$ canonically onto the universum $\mathbb{S}$ yields the behavior of the subsystem of $(\mathbb{U}, \mathcal{B})$ living in the universum $\mathbb{S}$.

We thus define **System** to be $2^{\mathbb{U}}$, ordered by reverse inclusion. Two behaviors $\mathcal{B}$ and $\mathcal{B}'$ then admit a least upper-bound (with respect to the reverse inclusion) corresponding to $\mathcal{B} \cap \mathcal{B}'$. The space **Phenome** of phenomes is $2^{\mathbb{S}}$, also ordered by reverse inclusion. The canonical projection $p : \mathbb{U} \to \mathbb{S}$ then lifts to a map π : **System** → **Phenome** sending $\mathcal{B}$ to $p(\mathcal{B})$.

Proposition 3.5.7. *The map π is a surjective veil.*

Proof. The map π is clearly surjective and order-preserving. The set $\{B : S \supseteq \pi(B)\}$ has $p^{-1}(S)$ as a *minimum* element with respect to $\supseteq$. □

Generative effects are typically sustained as:

$$\pi(\mathcal{B} \cap \mathcal{C}) \neq \pi(\mathcal{B}) \cap \pi(\mathcal{C}).$$

Although the changes in $\mathcal{C}$ are not directly applied onto the subsystem, they do affect the subsystem through the other parts that are now concealed. The veil π induces a map i : **Phenome** → **System** sending set S to $S \times \mathbb{S}'$. The map $i\pi$ is then a kernel operator that destroys all the potential for effects to occur due to restrictions of the system in $\mathbb{S}'$.

Concealing parameters.

For an additional interpretation of the example, let us suppose $\mathbb{U} = \mathbb{M} \times \mathbb{L}$. The subsystem in play (whose universum corresponds to $\mathbb{M}$) could represent manifest variables that are observable, and the rest (whose universum corresponds $\mathbb{L}$) could represent latent variables or parameters that aid *internally* in the workings of the system. As such, changes in the internal parameters of a system affect the manifest variables.

3.5.4 Concealing interdependence.

The previous example may be further enhanced to understand interdependence between components of an interconnected system. Let us again suppose that $\mathbb{U} = \mathbb{S} \times \mathbb{S}'$. The decomposition yields two surjective maps $p : \mathbb{U} \to \mathbb{S}$ and $p' : \mathbb{U} \to \mathbb{S}'$. The maps lift to separate veils π : **System** $\to 2^{\mathbb{S}}$ and π' : **System** $\to 2^{\mathbb{S}'}$ sending a behavior $\mathcal{B}$ to $p(\mathcal{B})$ and $p'(\mathcal{B}')$, respectively.

We then define **Phenome** to be $2^{\mathbb{S}} \times 2^{\mathbb{S}'}$. An element of **Phenome** is thus a set $S \times S'$ with $S \subseteq \mathbb{S}$ and $S' \subseteq \mathbb{S}'$. We finally define π : **System** $\to$ **Phenome** to be $p \times p'$ sending $\mathcal{B}$ to $p(\mathcal{B}) \times p(\mathcal{B}')$.

Proposition 3.5.8. *The map π is a surjective veil.*

Proof. The map π is clearly surjective and order-preserving. The set $\{B : S \times S' \supseteq p(B) \times p'(B)\}$ has $p^{-1}(S) \cap p'^{-1}(S')$ (i.e., $S \times S'$) as a *minimum* element with respect to $\supseteq$. □

And indeed, generative effects are typically sustained as:

$$\pi(\mathcal{B} \cap \mathcal{B}') \neq \pi(\mathcal{B}) \cap \pi(\mathcal{B})$$

The veil π induces a canonical inclusion map i : **Phenome** $\to$ **System**. The map $i\pi$ is a kernel operator that destroys any potential interdependence between the components $\mathbb{S}$ and $\mathbb{S}'$.

3.6 Forgetting characteristics.

The second source of generative effects consists of forgetting characteristics, or properties, from the given system. In such a setting, the space of phenomes tends to be larger than that of systems. Indeed, phenomes then comprise the systems in concern as well as systems non-necessarily satisfying the desired characteristic to be forgotten. Generative effects emerge from the potential of the characteristic to enhance the interconnected system.

We can forget characteristics of a system, by defining a bigger set **Phenome** containing **System**. Every element of **Phenome** can be seen as a potential system that is not forced to satisfy the forgotten characteristic. Every element of **Phenome** can then be treated as a partial observation of a system. Such a partial observation can then be completed into a system satisfying the desired forgotten characteristic. To this end, we require a map c : **Phenome** $\to$ **Phenome** such that:

$$c(p) \in \textbf{System} \text{ for all } p$$

and:

C.1. $p \leq c(p)$ for all p.

C.2. If $p \leq p'$, then $c(p) \leq c(p')$, for all p and p'.

C.3. $cc(p) = c(p)$ for all p.

Notice the duality between the C.i in this section and the K.i in the previous ones. First, the operator c sends a partial observation to one that *contains* it. Second, the operator preserves the relation among the observation. Third, the operator does not modify partial observations that are already systems. Again, an operator satisfying C.1, C.2 and C.3 is usually termed a closure operator. The property C.i coincides with property A.i in the contagion situation, whenever **Phenome** is 2^{Σ} for some set Σ.

The operator c can intuitively be expected to sustain generative effects whenever $c(p \vee p') \neq c(p) \vee c(p')$ for some p and p'. The forgotten characteristic then indeed plays a role in the interaction of systems, to enhance the resulting combined system.

Given a closure operator c on **Phenome**, the set **System** is identified with the set of fixed-points $\{p : c(p) = p\}$. We may then define a canonical inclusion ι : **System** $\to$ **Phenome**. Every closure operator gives rise to a veil:

Proposition 3.6.1. *The map ι is injective and order-preserving, and for every $p \in$* **Phenome***, the set $\{s : p \leq \iota(s)\}$ has a minimum element.*

Proof. The map ι is injective by definition. It is also clearly order-preserving. Finally, the system $c(p)$ is the minimal element of $\{s : p \leq \iota(s)\}$. □

Conversely, every injective veil induces a closure operator on P:

Proposition 3.6.2. *If ι :* **System** $\to$ **Phenome** *is an injective order-preserving map such that $\{s : p \leq \iota(s)\}$ has a minimum element for every p, then there exists a unique surjective map q :* **Phenome** $\to$ **System** *such that $q\iota$ is the identity on* **System***, and ιq is a closure operator on* **Phenome***.*

Proof. For every $p \in$ **Phenome**, define $q(p)$ to be the minimum element of $\{s : p \leq \iota(s)\}$. The map $q\iota$ is the identity map as $q\iota(s)$ is the minimum element of $\{s' : \iota(s) \leq \iota(s')\}$, namely s as ι is injective. The map q is then surjective. The map ιq is a closure operator as $\iota q(p)$ is the smallest element $\{\iota(s) : p \leq \iota(s)\}$. The requirements C.i can be easily checked. Uniqueness of q follows from Proposition 3.7.4(i). □

Finally, whether or not generative effects are sustained by the veil ι depends on the properties of the closure operator c.

Proposition 3.6.3. *If $s, s' \in$* **System***, then:*

$$\iota(s \vee_{\mathbf{System}} s') \neq \iota s \vee_{\mathbf{Phenome}} \iota s' \textit{ iff } c(\iota s \vee_{\mathbf{Phenome}} \iota s') \neq c(\iota s) \vee_{\mathbf{Phenome}} c(\iota s').$$

Proof. We have $c(\iota s) = \iota s$ for every system s. We also have $c(\iota s \vee_{\mathbf{Phenome}} \iota s') = c\iota(s \vee_{\mathbf{System}} s')$ for every s and s'. □

We next provide some example situations of forgetting characteristics.

3.6.1 Zooming into a deductive system.

We return to our contagion example. Rather than considering the collection of all possible systems, and their interaction, we may zoom in on one particular system. Indeed, a system in the contagion example is itself a closure operator over 2^Σ, and thus may pave the way to generative effects.

Let $f : 2^\Sigma \to 2^\Sigma$ be a map that satisfies A.1, A.2 and A.3. We define **Phenome** to be 2^Σ, the space of all configurations (of whether a node is infected or not) ordered by inclusion. The space **System** will consist of all the configurations allowable by the dynamics, namely the fixed-points $\{S : fS = S\} \subseteq 2^\Sigma$. As seen in Proposition 3.3.8, those fixed-points form a complete lattice, where every pair of admissible configuration admits a join. We then define ι to be the canonical inclusion **System** → **Phenome**, and get:

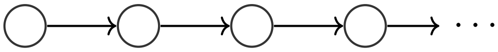
Proposition 3.6.4. *The map ι is an injective veil.*

Proof. The statement follows from Proposition 3.6.1. Indeed, the map ι is injective by definition. The system $f(S)$ is the minimal element of $\{s : S \leq \iota(s)\}$. □

Generative effects can sometimes be sustained, but not always. Whether or not the effects emerge depends on the properties of the closure operator f.

No generative effects.

Consider the following sequence of nodes.



A node becomes infected if a node pointing to it becomes infected. Once a node is infected, it remains infected forever. This system defines a map $f : 2^\Sigma \to 2^\Sigma$ satisfying A.1, A.2 and A.3, and induces a veil ι as described earlier.

Proposition 3.6.5. *The veil does not sustain generative effects.*

Proof. We have $f(S \cup S') = f(S) \cup f(S')$ for every S and S' in 2^Σ. □

This system by itself does not exhibit any cascading phenomenon. There might seem, however, to be an intuition for such a phenomenon only waiting to surface. Nevertheless, when restricted to this particular system, the observable part *prior to interaction* always determines the observable part *after the interaction*. The space of systems is linearly ordered in this case, consisting of a maximal chain in 2^Σ. To allow the intuition to reappear, one needs to enlarge the space of systems. The intuition of the phenomenon may be informally seen to come from the arcs of the directed graph. The arcs, however, are built into the situation, and cannot be modified. If we enlarge our space of systems to include some systems that may not have arcs among nodes, then we can recover generative effects. Enlarging so moves us one step closer to obtaining the whole class of maps satisfying A.1, A.2 and A.3 as a space of systems.

3.6.2 Causality in systems.

One may reconsider whether causality ought to be a concept of grandiose importance (see, e.g., [Rus12]). We view causality in this subsection as only intuitively expressing the notion of transitivity. A situation of cause and effect will be considered as a transitive relation $\to$ on a set Σ. The relation $a \to b$ can be interpreted as "a causes b". Transitivity then abuts to: if $a \to b$ and $b \to c$ then $b \to c$. Cause-and-effect seems to be inherent in cascade-like phenomena. They do appear, but only as a tangential special case of generative effects. The intuition arises once we decide to forget the property of transitivity from a relation.

We consider a system to be a transitive relation on Σ. The set **System** of transitive relations can be ordered by inclusion, and every pair of systems admits a least upper-bound in the poset. Two systems R and R' interact by taking the transitive closure $R \vee R'$ of their union $R \cup R'$. We can forget the transitivity property by *embedding* **System** in a greater lattice **Phenome** consisting of all relations on Σ. Two relations in **Phenome** interact by yielding their union. We define ι to be the canonical inclusion **System** $\to$ **Phenome**. Generative effects are sustained as, generally:

$$\iota(R \vee R') \neq \iota(R) \cup \iota(R').$$

The transitivity property plays a role in the interaction, leading to more causal relations than what would typically be expected without it. This generativity in the phenome is obtained by concealing characteristics in the systems. Indeed, ι induces a closure operator on Σ that sends a phenome to its transitive closure.

Incorporating time.

Time can be trivially incorporated by defining $\Sigma = E \times T$, where E is a set representing events, and T is an ordered set representing time. One can further impose restrictions where (e, t) may cause (e', t') only if $t \leq t'$. We will not dwell on developing such extensions in this chapter.

3.6.3 Algebraic constructions.

Closure operators abound in mathematics. Those trivially include linear spans, convex hulls, and topological closure. As an example, let G be an abelian group and suppose S is its underlying set. The space **System** will be the set $\mathbf{Sub}(G)$ of subgroups of G ordered by inclusion. The space **Phenome** will be the set 2^S of subsets of S, again, ordered by inclusion. Interaction in **System** is given by the linear span $+$ operator, while the interaction in **Phenome** is given by set-union. We can then define a closure operator on the set S, that sends subsets to the subgroup it generates. The closure operator then defines an injective veil $\iota : \mathbf{Sub}(G) \to 2^S$. Generative effects are sustained as, generally:

$$\iota(H + H') \neq \iota(H) \cup \iota(H')$$

The group axioms interact so as to produce more elements than what is only observed. The veil is actually forgetting the group structure of the system.

Forgetting might not create effects.

Suppose M is an R-module, and assume G is its underlying abelian group. The group G is obtained by forgetting the multiplicative R-action of M. Let $\iota : \mathbf{Sub}(M) \to \mathbf{Sub}(G)$ be the map that sends a submodule of M to its underlying abelian group. The map ι is an injective veil. Generative effects are however never sustained. Indeed, for all submodules N and N', we have:

$$\iota(N + N') = \iota(N) + \iota(N').$$

The ring action plays no role that affects the underlying abelian group of a module.

3.6.4 Universal grammar, languages and merge.

It is argued in [BC15]—and more generally through the minimalist program, see e.g. [Cho93], [Cho95] and related work—that the human ability of language universally arises from a single non-associative operation termed *merge*. For the illustrative purpose of this subsection, let us define a set Σ of words. Let $(\operatorname{free}\Sigma, \wedge)$ denote the free non-associative commutative algebra generated by the elements of Σ. We refer to an element of $\operatorname{free}\Sigma$ as a *sentence*. Due to non-associativity, a *sentence* is then not a linear concatenated string but rather a hierarchical object, a tree whose leaves are elements of Σ. As expounded in [BC15], language is fundamentally hierarchical and not associative, i.e., not concatenated (or not linear). When externalized, say through speech, this sentence tends to be made concatenated. A grammar is then seen as a subalgebra of $(\operatorname{free}\Sigma, \wedge)$. Such a subalgebra is thought to be generated by a set of *sentences*. The subalgebra can be ordered by inclusion to yield a join-semilattice $\big\langle \mathbf{Sub}\langle \operatorname{free}\Sigma, \wedge\rangle, \vee\big\rangle$. The join $g \vee g'$ of two subalgebras g and g' is the subalgebra generated by $\{s \wedge s' : s \in g \text{ and } s \in g'\}$.

These grammars possess an intuitive generative power, where the merge operator $\wedge$ interacts with sentences to form new ones. To capture it, we forget such a property. Formally, we define a(n injective) veil, that sends a grammar (a subalgebra) to its underlying language (a set):

$$\iota : \big\langle \mathbf{Sub}\langle \operatorname{free}\Sigma, \wedge\rangle, \vee\big\rangle \to \langle 2^{\operatorname{free}\Sigma}, \cup\rangle$$

Generative effects are sustained as, in general, we have:

$$\iota(g \vee g') \neq \iota(g) \cup \iota(g')$$

The discrepancy is caused by the effect of the merge operator. It is obtained by forgetting the characteristic that a grammar is equipped by such an operator, leaving only the underlying language. It captures the *generativity* of the grammars considered.

3.7 On arbitrary veils.

An arbitrary veil needs neither be surjective nor injective. Indeed, any combination of concealing mechanisms and forgetting properties can lead to an adequate veil. In general:

Proposition 3.7.1. *The composition of two veils is a veil.*

Proof. The property V.1 is preserved under composition. Let $\Phi_1 : S \to Q$ and $\Phi_2 : Q \to P$ be veils. Then, by V.2, for every $p \in P$, the set $\{q : p \leq \Phi_2(q)\}$ has a minimum element q_{min} and the set $\{s : q_{min} \leq \Phi_1(s)\}$ has a minimum element s_{min}. If $p \leq \Phi_2\Phi_1(s)$, then $q_{min} \leq \Phi_1(s)$, and $s_{min} \leq s$. The set $\{s : p \leq \Phi_2\Phi_1(s)\}$ then has s_{min} as a minimum element. □

A *converse* also holds: every possible veil arises from a combination of concealing mechanisms and forgetting characteristics.

Proposition 3.7.2. *If* $\Phi : \mathbf{System} \to \mathbf{Phenome}$ *is a veil, then* Φ *admits a factorization* $\Phi = \iota\pi$ *such that* $\pi : \mathbf{System} \to Q$ *is a surjective veil, and* $\iota : Q \to \mathbf{Phenome}$ *is an injective veil.*

Proof. Let Q be the image set $\{\Phi(s) : s \in \mathbf{System}\}$. The map Φ factors as $\iota\pi$ through Q with π surjective, ι injective and both order-preserving. As Φ is a veil, the set $\{s : p \leq \iota\pi(s)\}$ admits a minimum element for every p. The set $\{\pi(s) : p \leq \iota\pi(s)\}$ then admits a minimum element as π is order-preserving. As π is surjective, we get that $\{q \in Q : p \leq \iota q\}$ admits a minimum element for every p. Similarly, for every $q \in Q$, the set $\{s : \iota(q) \leq \iota\pi(s)\}$ admits a minimum element. As $Q \subseteq \mathbf{Phenome}$, we have $\iota(q) \leq \iota(q')$ if, and only if $q \leq q'$. The set $\{s : q \leq \pi(s)\}$ then admits a minimum element. □

The map π can be interpreted to conceal mechanisms in systems. The map ι can then be interpreted as further forgetting characteristics from the partially concealed systems. Every situation of generative effects arises from a combination of the two cases. A veil then sustain generative effects whenever one of its factor components sustains those effects.

Finally, a good way to recognize a veil is by checking whether it preserves meets (i.e., greatest lower-bounds):

Proposition 3.7.3. *If* **System** *and* **Phenome** *admit arbitrary meets, then: a map* $\Phi : \mathbf{System} \to \mathbf{Phenome}$ *is a veil if, and only if, it is order-preserving and preserves arbitrary meets.*

Proof. Suppose Φ is order-preserving and preserves arbitrary meets. The set $\{s : p \leq \Phi(s)\}$ contains the maximum element of **System** (the meet of the empty set) and is thus non-empty. The meet of all the elements in $\{s : p \leq \Phi(s)\}$ lies in this set and is its minimum element. Conversely, suppose Φ is a veil. Let $S \subseteq \mathbf{System}$ be a subset. We have $\Phi(\wedge S) \leq \wedge\Phi(S)$. Also, $\{s : \wedge\Phi(S) \leq \Phi(s)\}$ has a minimum element s_{min}. Obviously $s_{min} \leq s$ for every $s \in S$ and thus $s_{min} \leq \wedge S$. We then get $\wedge\Phi(S) \leq \Phi(\wedge S)$. □

In particular, Φ sends the maximum element in **System** to the maximum element of **Phenome**. Indeed, the greatest lower-bound for the empty set yields the maximum element in **Phenome**. Finally, if **System** and **Phenome** only admit finite meets, the veil would preserve them. Indeed, the converse direction of the proof above goes through unchanged (by only considering S to be finite) for the finite case.

3.7.1 Properties and Galois connections.

The property V.2 is crucial as it allows us to define a *freest* (or simplest) reconstruction of a system from the phenome. If $\Phi : \mathbf{System} \to \mathbf{Phenome}$ is a veil, then define $F : \mathbf{Phenome} \to \mathbf{System}$ (for free) such that:

$$F : p \mapsto \min\{s : p \leq \Phi(s)\}$$

If Φ is invertible, then F would be the inverse of Φ. In the cases of interest, Φ is not invertible, and F ought to be interpreted as the closest map that we could have as an inverse. The map F is said to be the **left** adjoint of Φ, and the map Φ is said to be the **right** adjoint of F. The pair (F, Φ) is termed a Galois connection. We provide some properties related to Galois connections, and refer the reader to [Bir67] Ch. V, [Eve44], [Ore44] and [EKMS93] for a thorough treatment. We will, however, not be explicitly using those properties.

Proposition 3.7.4. *Let Φ be a veil, and F be $p \mapsto \min\{s : p \leq \Phi(s)\}$, then:*

i. For all p and s, we have $F(p) \leq s$ iff $p \leq \Phi(s)$.

ii. The map $F\Phi$ is a kernel operator on **System**.

iii. The map ΦF is a closure operator on **Phenome**.

iv. The map Φ maps s to the maximum of $\{s : s \leq F(p)\}$.

v. For all p and p', we have $F(p \vee p') = F(p) \vee F(p')$.

Proof. We refer the reader to [EKMS93] for proof of those statements, as well as other related statements. □

The map F is the unique map such that the (*i.*) holds. Furthermore, item (*ii.*) recovers the kernel operator that conceals mechanisms in the system. Item (*iii.*) recovers the closure operator that forgets characteristics of the systems.

One important consequence is:

Corollary 3.7.5. *If $\Phi : (\mathbf{System}, \leq) \to (\mathbf{Phenome}, \leq)$ is a veil, then its adjoint $F : (\mathbf{Phenome}, \geq) \to (\mathbf{System}, \geq)$ defines a veil on the dual preordered sets.* □

Examples of Galois connections abound, especially when it comes to free construction of algebraic objects. Most are mathematical examples, but when interpreted appropriately yield us an intuition for cascade-like phenomena.

3.8 Dynamical generative effects.

We introduce, in this section, the notion of a dynamical veil. Such veils can be used to incorporate temporal information in the phenome. Aside from increasing modeling expressivity, dynamical veils can be used to *spread* generative effects over (temporal) approximations of systems. Such a spread may be used for a relative/successive analysis of generative effects. This last direction will not be pursued in this chapter.

A main argument in this chapter is that generative effects enclose the intuitive notion of cascade effects. The term *cascade* however gives an impression of an evolving process. The notion of time then seems to be an essential component for cascades. However generative effects do not depend intrinsically on time. Interconnection of systems does not depend on time either. The goal of this interlude section is to aid in reconciling this view. We thus introduce the notion of an $\mathcal{I}$-dynamical veil. Whenever generative effects are sustained by such a veil, we may think of them as dynamically realized.

Definition 3.8.1. *Let $\mathcal{I}$ be a preordered set. A veil $\Phi : \mathcal{S} \to \mathcal{P}$ is said to be an $\mathcal{I}$-dynamical veil if $\mathcal{P}$ is isomorphic to $P^{\mathcal{I}}$ (i.e., the preordered set of order-preserving maps $\mathcal{I} \to P$) for some preordered set P.*

We often consider, in this section, sets $\mathcal{I}$ that are linearly ordered, i.e., where $\leq$ is antisymmetric for every $i, j \in \mathcal{I}$, either $i \leq j$ or $j \leq i$. Such linear orders may be used to account for time, indexed by the elements of $\mathcal{I}$.

The notion of a system and the means of interconnecting systems remain unchanged. The phenome is then obtained by *reading* information from a system indexed by $\mathcal{I}$. The space of systems then needs to be rich in (e.g., temporal) structure to support a *meaningful* $\mathcal{I}$-dynamical veil.

3.8.1 Revisiting contagion, dynamically.

Let us reconsider the systems on contagion (or deduction) considered in Section 3.3 and studied in [Ada17c]. Let $\mathcal{I}$ be a preordered set, and let P be a complete lattice. We define $\mathcal{L}_{P^{\mathcal{I}}}$ to be the set of maps $f : P^{\mathcal{I}} \to P^{\mathcal{I}}$ satisfying:

A.1 If $a \in P$, then $a \leq f(a)$.

A.2 If $a \leq b$, then $f(a) \leq f(b)$.

A.3 If $a \in P$, then $f(f(a)) = f(a)$.

The set $\mathcal{L}_{P^{\mathcal{I}}}$ may be naturally ordered to form a lattice. We may then define an order-preserving map:

$$\text{eval} : \mathcal{L}_{P^{\mathcal{I}}} \to P^{\mathcal{I}}$$

sending a system to its least fixed-point. The map eval preserves arbitrary meets, and admits a left adjoint free : $P^{\mathcal{I}} \to \mathcal{L}_{P^{\mathcal{I}}}$. The map eval may also be defined to act on $\mathcal{L}_P$, the maps $P \to P$ satisfying A.1, A.2 and A.3, as done in Section 3.3 through Φ.

Syntax and interpretation.

We let Σ be a finite set of n elements. The set 2^Σ denotes the set of subsets (or powerset) or Σ. We set P to be 2^Σ, and consider $\mathcal{I}$ to be the (canonically) preordered set $\mathbb{Z}^{\geq 0}$ of non-negative integers.

A system $f \in \mathcal{L}_{P^\mathcal{I}}$ may then be syntactically described by a map $\mathcal{N} : \Sigma \to 2^{2^\Sigma \times \mathbb{Z}^{\geq 0}}$. Indeed, every element i of Σ is attributed a collection of pairs (S, d), where $S \subseteq \Sigma$ and $d \in \mathbb{Z}^{\geq 0}$. The interpretation is as follows. Let $X_0, X_1, X_2, \cdots$ be subsets of Σ where X_0 is the empty set. If S belongs to X_m, then $i \in X_{m+d}$. The rule of course applies *simultaneously* to all pairs (S, d) for a given i, and for every element i of Σ.

We may interpret X_m to denote the elements that are active (infected or failed) at time m. If the elements of S are already active (infected or failed) at time m (i.e., belong to X_m), then i will become active (infected or failed) after d time steps from m, i.e. at time $m + d$, belonging to X_{m+d}.

3.8.2 From dynamical veils to veils.

The phenome, in the case of an $\mathcal{I}$-dynamical veil, is thus a collection of related frames, or snapshots, taken from the system. In case $\mathcal{I}$ is a linearly ordered set, the frames are successive snapshots of the system. We can easily focus on one of the frames, forgetting others, and still recover a situation of generative effects.

Let $\mathcal{I}$ and P be preordered sets. We define $\pi_i : P^\mathcal{I} \to P$ to be the canonical projection onto the ith component of $\mathcal{I}$. Applying π_i on the phenome, a collection of frames (or snapshot), amounts to only keeping the ith frame (or snapshot).

Proposition 3.8.2. *If P is finitely complete (resp. finitely cocomplete) then for every $a, b \in P^\mathcal{I}$, we have $\pi_i(a \wedge b) = \pi_i(a) \wedge \pi_i(b)$ (resp. $\pi_i(a \vee b) = \pi_i(a) \vee \pi_i(b)$).*

Proof. Joins and meets in $P^\mathcal{I}$ are computed pointwise, if they exist in P. □

The projection π_i is often also a veil.

Proposition 3.8.3. *If P admits a minimum element, then π_i is a veil.*

Proof. The map π_i is clearly order-preserving. Let 0 be the minimum element of P. For every p in P, the map q^* sending j to p if $i \leq j$ and to 0 otherwise is the minimum of the set $\{q : p \leq \pi_i q\}$. □

This veil however does not sustain generative effects. This fact is a desirable feature. Indeed, by composing a veil $\Phi : \mathcal{S} \to P^\mathcal{I}$ with π_i we do not create additional generative effects. We may then analyze Φ by separately analyzing the π_i's.

There is also a more interesting means of recovering a single snapshot, achieved by aggregating everything. In such a case, we retrieve the *asymptotic* or *limiting* behavior. To this end, we define the map:

$$\text{colim} : P^\mathcal{I} \to P$$

that sends $a \in P^\mathcal{I}$ to $\vee_i a_i$.

The map colim is not always a veil. As an example, suppose $\mathcal{I} = \mathbb{Z}^{\geq 0}$ (canonically preordered) and $P = \{0, 1\}$ with $0 \leq 1$. Indeed, the set $\{a \in \{0,1\}^{\mathbb{Z}} : 1 \leq \operatorname{colim} a\}$ does not admit a minimum element. Regardless, the map colim is well behaved towards existing generative effects. Indeed:

Proposition 3.8.4. *The map* colim *preserves arbitrary joins, whenever they exist.*

Proof. Trivially $(\vee_i a_i) \vee (\vee_i b_i) = \vee_i (a_i \vee b_i)$ being the least upper-bound of $\{a_i\} \cup \{b_i\}$. □

We can thus aggregate phenomes, in the case of the dynamical contagion example over 2^Σ, without creating new generative effects. Let $\mathcal{I}$ be a preordered set and P be a complete lattice. Recall that $\mathcal{L}_{P^\mathcal{I}}$ (resp. $\mathcal{L}_P$) denotes the set of maps $P^\mathcal{I} \to P^\mathcal{I}$ (resp. $P \to P$) satisfying A.1, A.2 and A,3.

Proposition 3.8.5. *For $a \in P$, let $a^\mathcal{I}$ denote the constant map in $\mathcal{I} \to P$ with image a. If $f \in \mathcal{L}_{P^\mathcal{I}}$, then the map* $\operatorname{agg} f : P \to P$*:*

$$\operatorname{agg} f : a \mapsto \operatorname{colim} f(a^\mathcal{I})$$

belongs to $\mathcal{L}_P$.

Proof. (A.1) Since $a^\mathcal{I} \leq f(a^\mathcal{I})$ by A.1 of $\mathcal{L}_{P^\mathcal{I}}$, then $\operatorname{colim} a^\mathcal{I} \leq \operatorname{colim} f(a^\mathcal{I})$ as colim is order-preserving. (A.2) If $a \leq b$, then $a^\mathcal{I} \leq b^\mathcal{I}$. It then follows that $f(a^\mathcal{I}) \leq f(b^\mathcal{I})$ by A.2 of $\mathcal{L}_{P^\mathcal{I}}$, and thus $\operatorname{colim} f(a^\mathcal{I}) \leq \operatorname{colim} f(b^\mathcal{I})$. (A.3) Finally, $\operatorname{colim} f(\operatorname{colim} f(a^\mathcal{I}))^\mathcal{I} = \operatorname{colim}(\operatorname{colim} f(a^\mathcal{I}))^\mathcal{I} = \operatorname{colim} f(a^\mathcal{I})$ as f satisfies A.3 of $\mathcal{L}_{P^\mathcal{I}}$. □

In the case where $P = 2^\Sigma$ and $\mathcal{I} = \mathbb{Z}^{\geq 0}$, the map agg can be interpreted to send a system with syntactic description $\mathcal{N}$ to one where all pairs (S, d) for element i are replaced by $(S, 0)$. In general, we recover the following commutative situation:

Proposition 3.8.6. *If $\mathcal{I}$ is a preordered set and P is a complete lattice, then the diagram:*

$$\begin{tikzcd}
\mathcal{L}_{P^\mathcal{I}} \arrow[r, "\text{eval}"] \arrow[d, "\text{agg}"'] & P^\mathcal{I} \arrow[d, "\text{colim}"] \\
\mathcal{L}_P \arrow[r, "\text{eval}"] & P
\end{tikzcd}$$

commutes, i.e. $\operatorname{colim} \circ \operatorname{eval} = \operatorname{eval} \circ \operatorname{agg}$.

Proof. If 0 denote the minimum element of P, then $\operatorname{eval}(\operatorname{agg} f) = (\operatorname{agg} f)(0) = \operatorname{colim} f(0^\mathcal{I}) = \operatorname{eval} f$. □

We have been *projecting* the phenome in $P^\mathcal{I}$ to a phenome in P, while making sure that joins are preserved. Join-preservation then does not create new generative effects. The projection however will often remove some of the original generative effects. One ought to then think of such procedures as focusing on a particular refined aspect of the dynamical phenome.

3.8.3 From veils to dynamical veils.

Any veil is trivially a dynamical veil over the one-point poset. We can however obtain a veil that is non-trivially dynamical by considering the systems to be given by a *filtration.*

Definition 3.8.7. *Let $\mathcal{I}$ be a preordered set. An I-filtration of a system s in a preordered set $\mathcal{S}$ is an order-preserving map $F : \mathcal{I} \to \mathcal{S}$ such that* $\operatorname{colim} F = s$*.*

An $\mathcal{I}$-filtration then provides a *successive* approximation for the system s. Every veil $\Phi : \mathcal{S} \to \mathcal{P}$ induces a canonical veil $\Phi^{\mathcal{I}} : \mathcal{S}^{\mathcal{I}} \to \mathcal{P}^{\mathcal{I}}$. Defining a system by an $\mathcal{I}$-filtrations can be seen to equip it with adequate temporal information.

In the behavioral approach.

Through the lens of the behavioral approach to systems theory, such temporal information can be seen as a further refinement of constraints. Let $\mathcal{I}$ be a linearly ordered set of n elements, and consider the set $\mathbb{U} = U_1 \times \cdots \times U_n$. For every i, let $p_i : \mathbb{U} \to U_i$ be the canonical projection. For every Willems' system $(\mathbb{U}, \mathcal{B})$, the sets:

$$F_i\mathcal{B} = \pi_i^{-1}\pi_i\mathcal{B}$$

then define an $\mathcal{I}$-filtration of $\mathcal{B}$. Every set U_i can be seen to represent a variable, an $\mathcal{I}$-filtration can then be seen to successively *grow* the constraints to connect different variables.

3.8.4 Why care about dynamical veils?

Dynamical veils can be used to incorporate *time* in generative effects. There are, however, various other ways of incorporating time in generative effects, such as having the phenome contain timed trajectories. However, going from a veil to a dynamical veil, by resolving a system into an $\mathcal{I}$-filtration, may allow us to *spread* generative effects. The eventual goal fully developed in [Ada17g] and exemplified in [Ada17a], [Ada17e] and [Ada17d] is to develop *cohomology* theories for understanding such effects. These dynamical veils may allow us to develop *relative* theories. It other terms, it may allows to ask (and answer) the following informal question: suppose we have observed a cascade, and its effects, up to time T, what new effects resulting from the cascade will appear at time $T + m$? This direction will, however, not be further pursued in the chapter.

3.9 Factorization and lifts.

The second condition V.2 of a veil may seem to be restrictive. Some situations may be formalized in a way that does not yield such a condition. We show how one can recover a veil from non-veil-like situations.

3.9.1 Factoring and retrieving the intuition.

Let P and Q be cocomplete preordered sets (admitting arbitrary joins) and let $f : P \to Q$ be an order-preserving map. It can be the case that P and Q contain elements that make property V.2 fail for f, but that are irrelevant to any situation of generative effects possibly suggested by f. There exists a systematic way to get rid of such elements, and potentially retrieve a hidden veil.

On the P side.

Define a binary relation $\sim$ on P such that $p \sim q$ if, and only if, $f(p \vee x) = f(q \vee x)$ for all $x \in P$.

Proposition 3.9.1. *The relation $\sim$ is an equivalence relation.*

Proof. The relation can be trivially checked to be reflexive, symmetric and transitive. □

The relation $\sim$ is further compatible with the structure of P when viewed as a join semilattice.

Proposition 3.9.2. *The relation $\sim$ is a congruence relation on $(P, \vee)$, i.e., if $a \sim b$ and $a' \sim b'$ then $a \vee a' \sim b \vee b'$.*

Proof. Suppose $a \sim b$ and $a' \sim b'$. For every $x \in P$, we have:

$$f(a \vee a' \vee x) = f(b \vee (a' \vee x)) = f(a' \vee (b \vee x)) = f(b' \vee b \vee x).$$

The equalities follow from commutativity and associativity of $\vee$. □

To get a better understanding of $\sim$ we note:

Proposition 3.9.3. *If $a \leq c$ and $a \sim c$, then: for every b where $a \leq b \leq c$, we have $b \sim c$.*

Proof. If $a \leq b \leq c$, then for every x, we have $f(a \vee x) \leq f(b \vee x) \leq f(c \vee x)$. □

Proposition 3.9.4. *If $a \sim b$, then $b \sim a \vee b$.*

Proof. Indeed, we have that $a \sim b$ and $b \sim b$. The result then follows by congruence. Or directly, we have $f(a \vee b \vee x) = f(b \vee b \vee x) = f(b \vee x)$ □

In particular, a maximal element (if it exists) of a congruence class of $\sim$ is the (unique) maximum element of the class. The congruence classes induced by $\sim$ define a partition of P. These congruence classes inherit an order to yield a join-semilattice $P_\sim$. The order relation $\leq$ on $P_\sim$ is defined as $C \leq C'$ if, and only if, there is an $a \in C$ and $a \in C'$ such that $a \leq a'$. Equivalently, the join operation is defined as: if a and a' are in the classes C and C', then $C \vee C'$ is the congruence class containing $a \vee a'$. The join operation is well defined as $\sim$ is a congruence relation. Let $\pi : P \to P_\sim$ be the order-preserving surjective map that sends an element of P to its congruence class.

Proposition 3.9.5. *The map π commutes with finite joins.*

Proof. The statement follows by construction of $P_\sim$, namely from the fact that $\sim$ is a congruence relation. □

On the Q side.

Define $\hat{Q}$ to be sub-joinsemilattice of Q generated by $f(P)$. Namely we define $\hat{Q}$ to be the elements of $f(P)$ along with all possible finite joins ordered by the partial order of Q. Let $\iota : \hat{Q} \to Q$ be the canonical order-preserving injective map.

Proposition 3.9.6. *The map ι commutes with finite joins.*

Proof. The statement immediately follows from the construction of $\hat{Q}$. □

Combined.

The maps π and ι may be used to factorize f.

Proposition 3.9.7. *There exists a unique map $g : P_\sim \to \hat{Q}$ such that the diagram:*

$$\begin{tikzcd}
P_\sim \arrow[r, "g"] & \hat{Q} \arrow[d, "\iota"] \\
P \arrow[u, "\pi"] \arrow[r, "f"] & Q
\end{tikzcd}$$

commutes, i.e., $f = \pi g \iota$.

Proof. For every class C in $P_\sim$, let αC denote a fixed element of C. Define g to be $C \mapsto f(\alpha C)$. The diagram commutes, and the map g is unique as ι is injective and π is surjective. □

We then get:

Corollary 3.9.8. *For all $a, b \in P$, we have:*

$$f(a \vee b) \neq f(a) \vee f(b) \quad \textit{iff} \quad g(\pi a \vee_{P_\sim} \pi b) \neq g(\pi a) \vee_{\hat{Q}} g(\pi b).$$

Furthermore:

Proposition 3.9.9. *If $f : P \to Q$ is injective, then (i) π is the identity and, (ii) g is a veil if, and only if, for every p and p', we have:*

$$\textit{if } f(p) \leq f(p'), \textit{then } p \leq p'.$$

Proof. Let f be injective. Then $p \sim p'$ implies $p = p'$, and π is thus the identity.

Suppose that g is a veil. If $f(p) \leq f(p')$, then $g(p) \leq g(p')$. The greatest lower-bound of $g(p)$ and $g(p')$ then exists and is $g(p) \wedge g(p') = g(p)$. If we consider the set $T = \{t : g(p) \wedge g(p') \leq g(t)\}$, then it follows that $p, p' \in T$ and p is the (unique) minimum of T. We then have $p \leq p'$.

Conversely, suppose that $f(p) \leq f(p')$ implies $p \leq p'$ and consider the set $\{p : q \leq g(p)\}$ for $q \in \hat{Q}$. If $q \in f(P)$ then, as f is injective, the set $\{p : q \leq g(p)\}$ admits a unique minimum, the preimage of q with respect to f. If $q \notin f(P)$, then $q = \vee_i f(p_i)$ is a finite join of elements in $f(P)$. We also have:

$$\begin{aligned} \{p : \vee_i f(p_i) \leq g(p)\} &= \cap_i \{p : f(p_i) \leq g(p)\} \\ &= \cap_i \{p : f(p_i) \leq f(p)\} \\ &= \cap_i \{p : p_i \leq p\} \\ &= \{p : \vee_i p_i \leq p\} \end{aligned}$$

This set has $\vee_i p_i$ as a minimum element. □

Suppose that P is finite. Then, every congruence class C in $\sim$ admits a maximum element, the join of all its elements. We define $c : P \to P$ to be the map that sends an element p to the maximum element of its congruence class $\pi(p)$. Notice that c is a closure operator on P.

Proposition 3.9.10. *Let P be finite. If $f : P \to Q$ is surjective, then (i) ι is the identity, and (ii) g is a veil if, and only if, $f(\wedge_i p_i) = \wedge_i f(p_i)$ for every (finite) collection $\{p_i\} \subseteq c(P)$.*

Proof. If f is surjective, then $\hat{Q} = f(P) = Q$. The rest follows from Proposition 3.7.3. Indeed, $c(P)$ is isomorphic to $P_\sim$ and g is the restriction of f to $c(P)$. □

The finiteness condition can be alleviated through adequate technical care.

3.9.2 All order-preserving maps can be lifted to veils.

A factorization, as done in the previous section, need not always yield a veil. We will often have a map that is not necessarily a veil, but would still like to interpret the situation as one exhibiting generative effects. If we have a map that does not satisfy V.2, then some phenome will not have a minimum system that explains it. It will have multiple minimal systems explaining it. However if we can treat the multitude of systems as one *ambiguous* system, we can recover uniqueness.

In this section, we show that we can always lift an arbitrary order-preserving map to a veil between a lifted space of systems and a lifted space of phenomes. The relevant properties, namely whether or not it sustains generative effects, are preserved in the lift. We do lose something by this *completion*, as now the lifted space contains objects that we cannot necessarily interpret as systems. That need not be a nuisance as interaction of the interpretable systems is preserved. The generality of the lift can however restrict our ability to find tight structures for the situation.

Definition 3.9.11. *A filter (or upper set) J of a preordered set P is a subset of P such that: if $p \leq p'$ and $p \in J$, then $p' \in J$.*

In particular:

Proposition 3.9.12. *If P is finitely cocomplete, then: J is a filter of P if, and only if, $p \vee J \subseteq J$ for all $p \in P$.*

Proof. If J is a filter of P, then for every $j \in J$, we have $j \leq p \vee j$ and thus $p \vee j \in J$. Conversely, if $p \leq p'$ and $p \in J$, then $p' = p' \vee p \in p' \vee J \subseteq J$. □

We denote by $\mathcal{J}(P)$ the preordered set of filters of P. The set $(\mathcal{J}(P), \supseteq)$ ordered by reverse inclusion is necessarily a lattice, that admits arbitrary, joins (through set-intersection $\cap$) and meets (through set-union $\cup$). In particular, $\mathcal{J}(P)$ is a distributive lattice as $\cap$ and $\cup$ distribute over one another.

If $p \in P$, we define $\langle p \rangle$ to be the filter generated by p. Namely: $\langle p \rangle = \{p \vee a : a \in P\}$. The element $p \in P$ is then represented by $\langle p \rangle$ in $\mathcal{J}(P)$. Indeed:

Proposition 3.9.13. *If P is finitely cocomplete, then: for all $p, p' \in P$, we have $\langle p \vee p' \rangle = \langle p \rangle \cap \langle p' \rangle$.*

Proof. If $a \in \langle p \rangle \cap \langle p' \rangle$, then $p \vee p' \leq a$, and so $a \in \langle p \vee p' \rangle$. Conversely, we trivially have $\langle p \vee p' \rangle \subseteq \langle p \rangle \cap \langle p' \rangle$. □

Note that lifting to $\mathcal{J}(P)$ does not preserve meets. Meets however are not essential throughout the theory, they just happen to be a convenience.

Let $f : P \to Q$ be an order-preserving map. If I is a filter, then $f(I)$ does not have to be a filter. We define $\mathcal{J}(f) : \mathcal{J}(P) \to \mathcal{J}(Q)$ to be the map that sends a filter I to the filter closure $\langle f(I) \rangle$ of $f(I)$.

Proposition 3.9.14. *We have $\mathcal{J}(f)\langle p \rangle = \langle f(p) \rangle$ for all p.*

Proof. The set $f\langle p \rangle$ contains $f(p)$ as a minimum element. □

Furthermore:

Proposition 3.9.15. *The map $\mathcal{J}(f) : \langle \mathcal{J}(P), \supseteq \rangle \to \langle \mathcal{J}(Q), \supseteq \rangle$ is a veil.*

Proof. The map $\mathcal{J}(f)$ is clearly order-preserving. Furthermore, the set $\{I : J \supseteq \mathcal{J}(f)(I)\}$ has $\big\langle \{p : J \supseteq \langle fp \rangle\} \big\rangle$ as a *minimum* element with respect to $\supseteq$. □

Finally, the potential for generative effects is preserved by the lift.

Proposition 3.9.16. *If P and Q are finitely cocomplete, then:*

$$f(p \vee p') \neq f(p) \vee f(p') \quad \textit{iff} \quad \mathcal{J}(f)(\langle p \rangle \cap \langle p' \rangle) \neq \mathcal{J}(f)(\langle p \rangle) \cap \mathcal{J}(f)(\langle p' \rangle).$$

Proof. We have $f(p \vee p') \neq f(p) \vee f(p')$ if, and only if, $\langle f(p \vee p') \rangle \neq \langle f(p) \vee f(p') \rangle$. By Proposition 3.9.13, we get $\langle f(p) \vee f(p') \rangle = \langle f(p) \rangle \cap \langle f(p') \rangle$ and $\langle p \vee p' \rangle = \langle p \rangle \cap \langle p' \rangle$. The rest then follows by Proposition 3.9.14. □

The $\mathcal{J}$ operator however disregards any information on whether or not f already satisfies V.2. For instance, if f already satisfies V.2 we should expect that we ought not need a lot of elements to add in the lifted space while completing the space, if anything at all. Such *controlled* lifts can be achieved via the use of Grothendieck topologies on posets. Such a direction will not be pursued in this chapter. It is however an important direction to develop: it provides a solid path towards toposes and the theory of sheaves (see e.g. [AGV72]).

Remark.

Notice that if the spaces P and Q were not cocomplete, the lift will *create* joins in the lattice of filters.

3.10 Concluding remarks.

The development of generative effects has been carried out in the restrictive case of preordered sets. We can achieve greater generality, and higher expressiveness, by having our spaces of systems and phenomes be categories. We refer the reader to [Ada17g] for the details.

In the general level, the presence of generative effects can be linked to a loss of exactness. This loss can be recovered via methods in homological algebra. We can extract algebraic objects that encode a system's potential to produce generative effects. We can then use those objects to characterize the phenomenon, and link the behavior of the interconnected system to that of its separate systems. This direction is carried out in the coming chapters.

Chapter 4

Towards an algebra for cascade effects

Abstract

We introduce a new class of (dynamical) systems that inherently capture cascading effects (viewed as consequential effects) and are naturally amenable to combinations. We develop an axiomatic general theory around those systems, and guide the endeavor towards an understanding of cascading failure. The theory evolves as an interplay of lattices and fixed points, and its results may be instantiated to commonly studied *models* of cascade effects.

We characterize the systems through their fixed points, and equip them with two operators. We uncover properties of the operators, and express *global* systems through combinations of *local* systems. We enhance the theory with a notion of failure, and understand the class of shocks inducing a system to failure. We develop a notion of μ-rank to capture the energy of a system, and understand the minimal amount of *effort* required to fail a system, termed *resilience*. We deduce a dual notion of *fragility* and show that the combination of systems sets a limit on the amount of fragility inherited.

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4.1 Introduction

Cascade effects refer to situations where the expected behavior governing a certain system appears to be *enhanced* as this component is embedded into a greater system. The effects of change in a subsystem may pass through *interconnections* and enforce an indirect change on the state of any remote subsystem. As such effects are pervasive—appearing in various scenarios of ecological systems, communication infrastructures, financial networks, power grids and societal networks—there is an interest (and rather a need) to understand them. Models are continually proposed to capture instances of cascading behavior, yet the *universal* properties of this phenomenon remain untouched. Our goal is to capture some essence of cascade effects, and develop an axiomatic theory around it.

A reflection on such a phenomenon reveals two informal aspects of it. The first aspect uncovers a notion of *consequence* relation that seemingly drives the phenomenon. Capturing *chains of events* seems to be inescapably necessary. The second aspect projects cascade effects onto a theory of subsystems, combinations and interaction. We should not expect any cascading behavior to occur in *isolation*.

The line of research will be pursued within the context of systemic failure, and set along a guiding informal question. When handed a system of interlinked subsystems, when would a *small* perturbation in some subsystems induce the system to failure? The phenomenon of cascade effects (envisioned in this chapter) restricts the possible systems to those satisfying posed axioms. The analysis of cascade effects shall be perceived through an analysis on these systems.

We introduce a new class of (dynamical) systems that inherently capture cascading effects (viewed as *consequential* effects) and are naturally amenable to combinations. We develop a general theory around those systems, and guide the endeavor towards an understanding of cascading failure. The theory evolves as an interplay of lattices and fixed points, and its results may be instantiated to commonly studied *models* of cascade effects.

Our systems

The systems, in this introduction, will be motivated through an elementary example. This example is labeled M.0 and further referred to throughout the chapter.

M.0. *Let $G(V, A)$ be a digraph, and define $N(S) \subseteq V$ to be the set of nodes j with $(i, j) \in A$ and $i \in S$. A vertex is of one of two colors, either black or white. The vertices are initially colored, and X_0 denotes the set of black colored nodes. The system evolves through discrete time to yield $X_1, X_2, \cdots$ sets of black colored nodes. Node j is colored black at step $m+1$ if any of its neighbors i with $j \in N(i)$ is black at step m. Once a node is black it remains black forever.*

Our systems will consist of a collection of states along with internal dynamics. The collection of states is a finite set P. The dynamics dictate the evolution of the system through the states and are *governed* by a class of maps $P \to P$. The state space in M.0 is the set 2^V where each $S \subseteq V$ identifies a subset of *black* colored nodes; the dynamics are dictated by $g : X \mapsto X \cup N(X)$ as $X_{m+1} = gX_m$.

We intuitively consider some states to be *worse* or *less desirable* than others. The color *black* may be undesirable in M.0, representing a *failed* state of a node. State S is then considered to be worse than state T if it includes T. We formalize this notion by equipping P with a partial order $\leq$. The order is only partial as not every pair of states may be comparable. It is natural to read $a \leq b$ in this chapter as b is a worse (or less desirable) state than a. The state space 2^V in M.0 is ordered by set inclusion $\subseteq$.

We expect two properties from the dynamics driving the systems. We require the dynamics to be *progressive*. The system may only evolve to a state considered less desirable than its initial state. We also require *undesirability* to be preserved during an evolution. The less desirable the initial state of a system is, the less desirable the final state (that the system evolves to) will be. We force each map $f : P \to P$ governing the dynamics to satisfy two axioms:

A.1 If $a \in P$, then $a \leq fa$.

A.2 If $a, b \in P$ and $a \leq b$, then $fa \leq fb$.

The map $X \mapsto X \cup N(X)$ in M.0 satisfies both A.1 and A.2 as $S \subseteq S \cup N(S)$, and $S \cup N(S) \subseteq S' \cup N(S')$ if $S \subseteq S'$.

Our interest lies in the limiting outcome of the dynamics, and the understanding we wish to develop may be solely based on the *asymptotic* behavior of the system. In M.0, we are interested in the set X_m for m large enough as a function of X_0. As V is finite, it follows that $X_m = X_{|V|}$ for $m \geq |V|$. We are thus interested in the map $g^{|V|} : X_0 \mapsto X_{|V|}$. More generally, as iterative composition of a map satisfying A.1 and A.2 eventually yields idempotent maps, we equip the self-maps f on P with a third axiom:

A.3 If $a \in P$, then $ffa = fa$.

Our class of interest is the (self-)maps (on P) satisfying the axioms A.1, A.2 and A.3. Each system will be identified with one such map. The system generated from an instance of M.0 corresponds to the map $X_0 \mapsto X_{|V|}$.

The axioms A.1, A.2 and A.3 naturally permeate a number of areas of mathematics and logic. Within metamathematics and (universal) logic, Tarski introduced these three axioms (along with supplementary axioms) and launched his theory of consequence operator (see [Tar36] and [Tar56]). He aimed to provide a general characterization of the notion of deduction. As such, if S represents a set of statements taken to be true (i.e. premises), and $Cn(S)$ denotes the set of statements that can be *deduced* to be true from S, then Cn (as an operator) obeys A.1, A.2 and A.3. Many familiar maps also adhere to the axioms. As examples, we may consider the function that maps (i) a subset of a topological space to its topological closure, (ii) a subset of a vector space to its linear span, (iii) a subset of an algebra (e.g. group) to the subalgebra (e.g. subgroup) it generates, (iv) a subset of a Euclidean n-space to its convex hull. Such functions may be referred to as *closure operators* (see e.g. [Bir36], [Bir67], [Ore43] and [War42]), and are typically objects of study in *universal algebra*.

Goal and Contribution of the Chapter

This chapter has three goals. The first is to introduce and motivate the class of systems. The second is to present some properties of the systems, and develop preliminary tools for the analysis. The third is to construct a setup for cascading failure, and illustrate initial insight into the setup. The chapter will not deliver an exhaustive exposition. It will introduce the concepts and augment them with enough results to allow further development.

We illustrate the contribution through M.0. We define f and g to be the systems derived from two instances (V, A) and (V, A') of M.0.

We establish that our systems are uniquely identified with their set of fixed points. We can reconstruct f knowing only the sets S containing $N(S)$ (i.e. the fixed points of f) with no further information on (V, A). We further provide a complete characterization of the systems through the fixed points. The characterization yields a remarkable conceptual and analytical simplification in the study.

We equip the systems with a lattice structure, uncover operators ($+$ and $\cdot$) and express *complex* systems through formulas built from *simpler* systems. The $+$ operator *combines* the effect of systems, possibly derived from different models. The system $f + g$, as an example, is derived from $(V, A \cup A')$. The $\cdot$ operator *projects* systems onto each other allowing, for instance, the recovery of *local* evolution rule. We fundamentally aim to extract properties of $f+g$ and $f \cdot g$ through properties of f and g separately. We show that $+$ and $\cdot$ lend themselve to well behaved operations when systems are represented through their fixed-points.

We realize the systems as interlinked components and formalize a notion of *cascade effects*. Nodes in V are identified with maps $e_1, \cdots, e_{|V|}$. The system $f \cdot e_i$ then defines the evolution of the color of node i as a function of the system state, and is identified with the set of nodes that *reach* i in (V, A).

We draw a connection between shocks and systems, and enhance the theory with a notion of failure. We show that minimal shocks (that fail a system h) exhibit a unique property that uncovers *complement* subsystems in h, termed *weaknesses*. A system is shown to be *injectively* decomposed into its *weaknesses*, and any weakness in $h + h'$ cannot result but from the combination of weaknesses in h and h'.

We introduce a notion of μ-rank of a system—akin to the (analytic) notion of a norm as used to capture the energy of a system—and show that such a notion is unique should it adhere to natural principles. The μ-rank is tied to the number of connected components in (V, A) when A is symmetric.

We finally set to understand the minimal amount of *effort* required to fail a system, termed *resilience*. Weaknesses reveal a dual (equivalent) quantity, termed *fragility*, and further puts resilience and μ-rank on comparable grounds. The fragility is tied to the size of the largest connected component in (V, A) when A is symmetric. It is thus possible to formally define *high ranked* systems that are not necessarily *fragile*. The combination of systems sets a limit on the amount of fragility the new system inherits. Combining two subsystems cannot form a fragile system, unless one of the subsystems is initially fragile.

Outline of the chapter.

Section 2 presents mathematical preliminaries. We characterize the systems in Section 3, and equip them with the operators in Section 4. We discuss component realization in Section 5, and derive properties of the systems lattice in Section 6. We discuss cascade effects in Section 7, and provide connections to formal methods in Section 8. We consider cascading failure and resilience in Section 9, and conclude with some remarks in Section 10.

4.2 Mathematical preliminaries.

A partially ordered set or poset $(P, \leq)$ is a set P equipped with a (binary) relation $\leq$ that is reflexive, antisymmetric and transitive. The element b is said to cover a denoted by $a \prec b$ if $a \leq b$, $a \neq b$ and there is no c distinct from a and b such that $a \leq c$ and $c \leq b$. A poset P is graded if, and only if, it admits a rank function ρ such that $\rho(a) = 0$ if a is minimal and $\rho(a') = \rho(a) + 1$ if $a \prec a'$. The poset $(P, \leq)$ is said to be a lattice if every pair of elements admits a greatest lower bound (meet) and a least upper bound (join) in P. We define the operators $\wedge$ and $\vee$ that sends a pair to their meet and join respectively. The structures $(P, \leq)$ and $(P, \wedge, \vee)$ are then isomorphic. A lattice is distributive if, and only if, $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b and c. The pair (a, b) is said to be a modular pair if $c \vee (a \wedge b) = (c \vee a) \wedge b$ whenever $c \leq b$. A lattice is modular if all pairs are modular pairs. Finally, a *finite* lattice is (upper) semimodular if, and only if, $a \vee b$ covers both a and b, whenever a and b cover $a \wedge b$.

Notation

We denote $f(g(a))$ by fga, the composite ff by f^2, and the inverse map of f by f^{-1}. We also denote $f(i)$ by f_i when convenient.

4.3 The Class of systems.

The state space is taken to be a *finite* lattice $(P, \leq)$. We consider in this chapter only posets $(P, \leq)$ that are lattices, as opposed to arbitrary posets. It is natural to read $a \leq b$ in this chapter as b is a worse (or less desirable) state than a. The meet (glb) and join (lub) of a and b will be denoted by $a \wedge b$ and $a \vee b$ respectively. A minimum and maximum element exist in P (by finiteness) and will be denoted by $\check{p}$ and $\hat{p}$ respectively.

A system is taken to be a map $f : P \to P$ satisfying:

A.1 If $a \in P$, then $a \leq fa$.

A.2 If $a, b \in P$ and $a \leq b$, then $fa \leq fb$.

A.3 If $a \in P$, then $ffa = fa$.

The set of such maps is denoted by $\mathcal{L}_P$ or simply by $\mathcal{L}$ when P is irrelevant to the context. This set is necessarily finite as P is finite.

Note on finiteness.

Finiteness is not essential to the development in the chapter; completeness can be used to replace finiteness when needed. We restrict the exposition in this chapter to finite cases to ease non-necessary details. As every finite lattice is complete, we will make no mention of completeness throughout.

4.3.1 Models and examples.

The axioms A.1 and A.2 hold for typical "models" adopted for cascade effects. We present three models (in addition to M.0 provided in Section 1) supported on the Boolean lattice, two of which—M.1 and M.3—are standard examples (see [Gra78], [Kle07] and [Mor00]). It can be helpful to identify a set 2^S with the set of all *black* and *white* colorings on the objects of S. A subset of S then denotes the objects colored *black*. The model M.1 generalizes M.0 by assigning *thresholds* to nodes in the graph. Node i is colored *black* when the number of neighbors colored *black* surpasses its threshold. The model M.2 is *noncomparable* to M.0 and M.1, and the model M.3 generalizes all of M.0, M.1 and M.2.

M.1. *Given a digraph over a set S or equivalently a map $N : S \to 2^S$, a map $k : S \to \mathbb{N}$ and a subset X_0 of S, let $X_1, X_2, \cdots$ be subsets of S recursively defined such that $i \in X_{m+1}$ if, and only if, either $|N_i \cap X_m| \geq k_i$ or $i \in X_m$.*

M.2. *Given a collection $\mathcal{C} \subseteq 2^S$ for some set S, a map $k : \mathcal{C} \to \mathbb{N}$ and a subset X_0 of S, let $X_1, X_2, \cdots$ be subsets of S recursively defined such that $i \in X_{m+1}$ if, and only if, either there is a $C \in \mathcal{C}$ containing i such that $|C \cap X_m| \geq k_c$ or $i \in X_m$.*

M.3. *Given a set S, a collection of monotone maps ϕ_i (one for each $i \in S$) from 2^S into $\{0, 1\}$ (with $0 < 1$) and a subset X_0 of S, let $X_1, X_2, \cdots$ be subsets of S recursively defined such that $i \in X_{m+1}$ if, and only if, either $\phi_i(X_m) = 1$ or $i \in X_m$.*

We necessarily have $X_{|S|} = X_{|S|+1}$ in the three cases above, and the map $X_0 \mapsto X_{|S|}$ is then in $\mathcal{L}_{2^S}$. The dynamics depicted above may be captured in a more general form.

M.4. *Given a finite lattice L, an order-preserving map $h : L \to L$, and $x_0 \in L$, let $x_1, x_2, \cdots \in L$ be recursively defined such that $x_{m+1} = x_m \vee h(x_m)$.*

We have $x_{|L|} = x_{|L|+1}$ and the map $x_0 \mapsto x_{|L|}$ is in $\mathcal{L}_L$.

The axioms allow greater variability if the state space is modified or augmented accordingly. Nevertheless, this chapter is only concerned with systems of the above form.

Note on realization.

Modifications of instances of M.i (e.g. altering values of k in M.1) may not alter the system function. As the interest lies in understanding *universal* properties of final evolution states, the analysis performed should be invariant under such modifications. However, analyzing the systems directly through their *form* (as specified through M.0, M.1, M.2 and M.3) is bound to rely heavily on the representation used. Introducing the axioms and formalism enables an understanding of systems that is independent of their representation. It is then a separate question as to whether or not a system may be realized through some form, or whether or not restrictions on form translate into interesting properties on systems. Not all systems supported on the Boolean lattice can be realized through the form M.0, M.1 or M.2. However, every system in $\mathcal{L}_{2^S}$ may be realized through the form M.3. Indeed, if $f \in \mathcal{L}_{2^S}$, then for every $i \in S$ define $\phi_i : 2^S \to \{0, 1\}$ where $\phi_i(a) = 1$ if, and only if, $i \in f(a)$. The map ϕ_i is monotone as f satisfies A.2. Realization is further briefly discussed in Section 4.5.

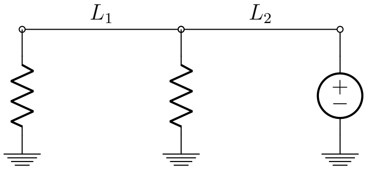
4.3.2 Context, interpretation and more examples.

A more *realistic* interpretation of the models M.i comes from a more realistic interpretation of the state space. This work began as an endeavor to understand the mathematical structure underlying models of diffusion of behavior commonly studied in the social sciences. The setup there consists of a population of interacting agents. In a societal setting, the agents may refer to individuals. The interaction of the agents affect their behaviors or opinions. The goal is to understand the spread of a certain behavior among agents given certain interaction patterns. Threshold models of behaviors (captured by M.0, M.1, M.2 and M.3) have appeared in the work of Granovetter [Gra78], and more recently in [Mor00]. Such models are key models in the literature, and have been later considered by computer scientists, see. e.g., [Kle07] for an overview.

The model described by M.1 is known as the linear threshold model. An individual adopts a behavior, and does not change it thereafter, if at least a certain number of its neighbors adopts that behavior. Various variations can also be defined, see e.g. M.2 and M.3, and again [Kle07] for an overview. The cascading intuition in all the variations however remains unchanged. These models can generally be motivated through a game theoretic setup. We will not be discussing such setups in this chapter. The *no-recovery* aspect of the models considered may be further relaxed by introducing appropriate time stamps. One such connection is described in [Kle07]. We are however interest in the instances where no-recovery occurs.

The models may also be given an interpretation in epidemiology. Every agent may either be healthy or infected. Interaction with an infected individual causes infections. This is in direct resemblance to M.0. Stochastics can also be added, either for a realistic approach or often for tractability. There is also a vast literature on processes over graphs, see e.g., [Dur07] and [New10]. Our aim is to capture the consequential effects that are induced by the interaction of several entities. We thus leave out any stochatics for the moment; they may be added later with technical work.

On a different end, inspired by cascading failure in electrical grids, consider the following simple resistive circuit. The intent is to guide the reader into a more realistic direction.



If line L_2 is disconnected from the voltage source, then line L_1 will also be disconnected from the source. Indeed, the current passing through L_1 has to pass through L_2. The converse is, of course, not true. This interdependence between L_1 and L_2 is easily captured by a system in $\mathcal{L}_{2^{\{L_1,L_2\}}}$. More general dependencies (notably failures caused by a redistribution of currents) can be captured, and concurrency can be taken care of by going to power sets. Indeed, M.0 also captures general reachability problems, where a node depicts an element of the state space. Specifically, let S be a set of states of some system, and consider a reflexive and transitive relation $\to$ such that $a \to b$ means that state b is reachable from state a. The map $2^S \to 2^S$ where $A \mapsto \{b : a \to b \text{ for some } a \in A\}$ satsfies A.1, A.2, and A.3 when 2^S is ordered by inclusion.

This work abstracts out the essential properties that gives rise to these situations. The model M.3 depicts the most general form over the boolean lattice. In M.3, the set S can be interpreted to contain n events, and an element of 2^S then depicts which events have occured. A system is then interpreted as a collection of (monotone) implications: if such and such event occurs, then such event occurs. The more general model M.4 will be evoked in Section 4.8, while treating connections to formal methods and semantics of programming languages.

On closure operators.

As mentioned in the introduction, the maps satisfying A.1, A.2 and A.3 are often known as closure operators. On one end, they appeared in the work of Tarski (see e.g., [Tar36] and [Tar56]). On another end, they appeared in the work of Birkhoff, Ore and Ward (see e.g., [Bir36], [Ore43] and [War42], respectively). The first origin reflects the consequential relation in the effects considered. The second origin reflects the theory of interaction of multiple systems. Closure operators appear as early as [Moo10]. They are intimately related to Moore families or closure systems (i.e., collection of subsets of S containing S and closed under intersection) and also to Galois connections (see e.g., [Bir67] Ch. V and [Eve44]). Every closure operator corresponds to a Moore family (see e.g., Subsection 4.3.3). This connection will be extensively used throughout the chapter. Most of the properties derived in Sections 4.3 and 4.4 can be seen to appear in the literature (see e.g. [Bir67] Ch. V and [CM03] for a recent survey). They are very elementary, and will be easily and naturally rederived whenever needed. Furthermore, every Galois connection induces one closure operator, and every closure operator arises from at least one Galois connection. Galois connection will be briefly discussed in Section 4.7. They will not however play a major explicit role in this chapter.

4.3.3 The fixed points of the systems.

As each map in $\mathcal{L}$ sends each state to a respective fixed point, a grounded understanding of a system advocates an understanding of its fixed points. We develop such an understanding in this subsection, and characterize the systems through their fixed points. Let Φ be the map $f \mapsto \{a : fa = a\}$ that sends a system to its set of fixed points.

Proposition 4.3.1. *If $f \neq g$ then $\Phi f \neq \Phi g$.*

Proof. If $\Phi f = \Phi g$, then $ga \leq gfa = fa$ and $fa \leq fga = ga$ for each a. Therefore $f = g$. □

It is obvious that each state is mapped to a fixed point; it is less obvious that, knowing only the fixed points, the system can be reconstructed uniquely. It seems plausible then to directly define systems via their fixed point, yet doing so inherently supposes an understanding of the image set of Φ.

Proposition 4.3.2. *If $f \in \mathcal{L}_P$, then $\hat{p} \in \Phi f$.*

Proof. Trivially $\hat{p} \leq f\hat{p} \leq \hat{p}$. □

Furthermore,

Proposition 4.3.3. *If $a, b \in \Phi f$, then $a \wedge b \in \Phi f$.*

Proof. It follows from A.2 that $f(a \wedge b) \leq fa$ and $f(a \wedge b) \leq fb$. If $a, b \in \Phi f$, then $f(a \wedge b) \leq fa \wedge fb = a \wedge b$. The result follows as $a \wedge b \leq f(a \wedge b)$. □

In fact, the properties in Propositions 4.3.2 and 4.3.3 fully characterize the image set of Φ.

Proposition 4.3.4. *If $S \subseteq P$ is closed under $\wedge$ and contains $\hat{p}$, then $\Phi f = S$ for some $f \in \mathcal{L}_P$.*

Proof. Construct $f : a \mapsto \inf\{b \in S : a \leq b\}$. Such a function is well defined and satisfies A.1, A.2 and A.3. □

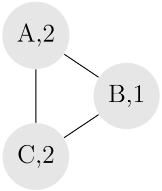
It follows from Propositions 4.3.2 and 4.3.3 that Φf forms a lattice under the induced order $\leq$. This conclusion coincides with that of Tarski's fixed point theorem (see [Tar55]). However, one additional structure is gained over arbitrary order-preserving maps. Indeed, the meet operation of the lattice $(\Phi f, \leq)$ coincides with that of the lattice $(P, \leq)$.

Example 4.3.5. *Let $f : 2^V \to 2^V$ be the system derived from an instance (V, A) of M.0. The fixed points of f are the sets $S \subseteq V$ such that $S \supseteq N(S)$. If S and T are fixed points of f, then $S \cap T$ is a fixed point of f. Indeed, the set $S \cap T$ contains $N(S \cap T)$. The map f sends each set T to the intersection of all sets $S \supseteq T \cup N(S)$. Although every collection C of sets in 2^V closed under $\cap$ and containing V can form a system, it will not always be possible to find a digraph where C coincides with the sets $S \supseteq N(S)$. The model M.0 is not* complex *enough to capture all possible systems.*

The space $\mathcal{L}$ is thus far only a set, with no further mathematical structure. The theory becomes lively when elements of $\mathcal{L}$ become *related*.

4.3.4 Overview through an example.

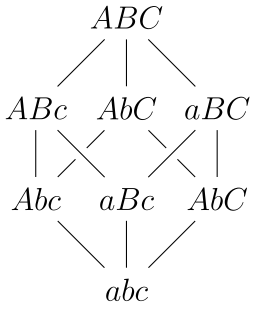
We illustrate some main ideas of the chapter through an elementary example. The example will run throughout the chapter, revisited in each section to illustrate its corresponding notions and results. The example we consider is the following (undirected) instance of M.1:



The nodes are labeled A, B and C. Each node I is tagged with an integer k_I that denotes a *threshold*. Each node can then be in either one of two colors: *black* or *white*. Node I is colored black (and stays black forever) when at least k_I neighbors are *black*. In our example, node A (resp. C) is colored black when both B and C

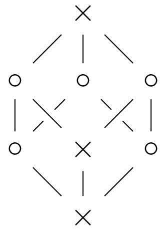
(resp. A) are black. Node B is colored black when either A or C are black. A node remains white otherwise.

The set underlying the state space is the set of possible colorings of nodes. Each coloring may be identified with a subset of $\{A, B, C\}$ containing the black colored nodes. The state space will then be identified with $2^{\mathbf{3}}$, the set of all subsets of $\{A, B, C\}$. The set $2^{\mathbf{3}}$ admits a natural ordering by inclusion ($\subseteq$) that turns it into a lattice. It may then be represented through a *Hasse diagram* as:



Notation: We denote subsets of $\{A, B, C\}$ as strings of letters. Elements in the set are written in uppercase, while elements not in the set are written in lowercase. Thus aBC, Abc and abc denote $\{B, C\}$, $\{A\}$ and $\{\}$ respectively. The string AC (with b/B absent) denotes both AbC and ABC.

The system derived from our example is the map $f : 2^{\mathbf{3}} \to 2^{\mathbf{3}}$ satisfying A.1, A.2 and A.3 such that $A \mapsto ABC$, $C \mapsto ABC$ and all remaining states are left unchanged. The fixed points of f yield the following representation.



We indicate, on the diagram, a fixed point by × and a non-fixed point by ∘.

4.3.5 On the system maps and their interaction.

As mentioned in the introduction, the systems of interest consist of a collection of states along with internal dynamics. The collection of states is a finite set P. The dynamics dictate the evolution of the system through the states and are *governed* by a class $\mathcal{K}$ of maps $P \to P$. The class $\mathcal{K}$ is closed under composition, contains the identity map and satisfies:

P.1 If $a \neq b$ and $fa = b$ for some $f \in \mathcal{K}$, then $gb \neq a$ for every $g \in \mathcal{K}$.

P.2 If $gfa = b$ for some $f, g \in \mathcal{K}$, then $hga = b$ for some $h \in \mathcal{K}$.

The principles P.1 and P.2 naturally induce a partial order $\leq$ on the set P. The principles P.1 and P.2 further force the functions to be *well adapted* to this order.

Proposition 4.3.6. *There exists a partial order $\leq$ on P such that for each $f \in \mathcal{K}$:*

A.1 If $a \in P$, then $a \leq fa$.

A.2 If $a, b \in P$ and $a \leq b$, then $fa \leq fb$.

Proof. Define a relation $\leq$ on P such that $a \leq b$ if, and only if, $b = fa$ for some $f \in \mathcal{K}$. The relation $\leq$ is reflexive and transitive as $\mathcal{K}$ is closed under composition and contains the identity map, respectively. Both antisymmetry and A.1 follow from P.1. Finally, if $a \leq b$, then $b = ga$ for some g. It then follows by P.2 that $fb = fga = hfa$ for some h. Therefore, $fa \leq fb$. □

We only alluded that the maps in $\mathcal{K}$ will govern our dynamics. No law of interaction is yet specified as to *how* the maps will govern the dynamics. As the state space is finite, the interaction may be motivated by iterative (functional) composition. For some map $\phi : \mathbb{N} \to \mathcal{K}$, the system starts in a state a_0 and evolves through $a_1, a_2, \cdots$ with $a_{i+1} = \phi_i a_i$. We reveal properties of such an interaction.

Let $\phi : \mathbb{N} \to \mathcal{S} \subseteq \mathcal{K}$ be a surjective map, and define a map F_i recursively as $F_1 = \phi_1$ and $F_{i+1} = \phi_{i+1} F_i$.

Proposition 4.3.7. *For some M, we have $F_m = F_M$ for $m \geq M$.*

Proof. It follows from A.1 that $F_1 a \leq F_2 a \leq \cdots$. The result then follows from finiteness of P. □

Proposition 4.3.8. *The map F_M is idempotent if $\phi^{-1} f$ is a non-finite set for each $f \in \mathcal{S}$.*

Proof. If $\phi^{-1} f$ is non-finite, then $fF = F$. If $\phi^{-1} f$ is non-finite for all $f \in \mathcal{S}$, then $FF = F$ as F is the finite composition of maps in $\mathcal{S}$. □

Let $\psi : \mathbb{N} \to \mathcal{S}$ be another surjective map, and define a map G_i recursively as $G_1 = \phi_1$ and $G_{i+1} = \phi_{i+1} G_i$. For some N, we necessarily get $G_N = G_n$ for $n \geq N$.

Proposition 4.3.9. *It follows that $F_M = G_N$, if $\phi^{-1} f$ and $\psi^{-1} f$ are non-finite sets for each $f \in \mathcal{S}$.*

Proof. Define $F = F_M$ and $G = G_N$. As F and G are idempotent, then $FG = G$ and $GF = F$. Therefore $Fa \leq FGa = Ga$ and $Ga \leq GFa = Fa$. □

The maps governing the dynamics are to be considered as *intrinsic mechanism wired into* the system. The effect of each map should not die out along the evolution of the system, but should rather keep on resurging. Such a consideration hints to an interaction insisting each map to be applied infinitely many times. There is immense variability in the order of application. However, we only want to care about the limiting outcome of the dynamics. By Proposition 4.3.9, such a variability would then make no difference from our standpoint. We further know, through Proposition 4.3.8, that iterative composition in this setting cannot lead but to idempotent maps. We then impose—with no loss in generality—a third principle (P.3) on $\mathcal{K}$ to contain only idempotent maps. This principle gives rise to a third axiom.

A.3 For $a \in P$, $ffa = fa$.

We define $\mathcal{L}_P$ to be the set of maps satisfying A.1, A.2 and A.3. The set $\mathcal{L}_P$ is closed under composition and contains each element of $\mathcal{K}$ with P.3 imposed, including the identity map. Furthermore, the principles P.1, P.2 and P.3 remain satisfied if $\mathcal{K}$ is replaced by $\mathcal{L}_P$. We will then extend $\mathcal{K}$ to be equal to $\mathcal{L}_P$. This extension offers greater variability in dynamics, and there is no particular reason to consider any different set. We further consider only posets $(P, \leq)$ that are lattices, as opposed to arbitrary posets.

4.4 The lattice of systems.

The theory of cascade effects presented in this chapter is foremost a theory of combinations and interconnections. As such, functions shall be treated in relation to each other. The notion of desirability on states introduced by the partial order translates to a notion of desirability on systems. We envision that systems combined together should form less desirable systems, i.e. systems that more likely to evolve to less desirable states. Defining an order on the maps is natural to formalize such an intuition. We define the relation $\leq$ on $\mathcal{L}$, where $f \leq g$ if, and only if, $fa \leq ga$ for each a.

Proposition 4.4.1. *The relation $\leq$ is a partial order on $\mathcal{L}$, and the poset $(\mathcal{L}, \leq)$ is a lattice.*

Proof. The reflexivity, antisymmetry and transitivity properties of $\leq$ follow easily from A.1 and A.2. If $f, g \in \mathcal{L}$, then define $h : a \mapsto fa \wedge ga$. It can be checked that $h \in \mathcal{L}$. Let h' be any lower bound of f and g, then $h'a \leq fa$ and $h'a \leq ga$. Therefore $h'a \leq fa \wedge ga = ha$, and so every pair in $\mathcal{L}$ admits a greatest lower bound in $\mathcal{L}$. Furthermore, the map $a \mapsto \hat{p}$ is a maximal element in $\mathcal{L}$. The set of upper bounds of f and g in $\mathcal{L}$ is then non-empty, and necessarily contains a least element by finiteness. Every pair in $\mathcal{L}$ then also admits a least upper bound in $\mathcal{L}$. □

We may then deduce join and meet operations denoted by $+$ (combine) and $\cdot$ (project) respectively. The meet of a pair of systems was derived in the proof of Proposition 4.4.1.

Proposition 4.4.2. *If $f, g \in \mathcal{L}$, then $f \cdot g$ is $a \mapsto fa \wedge ga$.*

On a dual end,

Proposition 4.4.3. *If $f, g \in \mathcal{L}$, then $f + g$ is the least fixed point of the map $h \mapsto (fg)h(fg)$. As P is finite, it follows that $f + g = (fg)^{|P|}$.*

Proof. Define $h_0 = (fg)^{|P|}$. Since the map fg satisfies A.1 and A.2, then h_0 satisfies A.1 and A.2. Furthermore, iterative composition yields $(fg)^{|P|+1} = (fg)^{|P|}$. Then h_0 is idempotent i.e. satisfies A.3. The map h_0 is then a fixed point of $h \mapsto (fg)h(fg)$. Moreover, every upperbound on f and g is a fixed point of $h \mapsto (fg)h(fg)$. Let h' be such an upperbound, then $fh' = h'$ and $gh' = h'$. It follows that $h_0 h' = h'$ i.e. $h_0 \leq h'$. □

The lattice $\mathcal{L}_P$ has a minimum and a maximum as it is finite. The minimum element (denoted by 0 or 0_p) corresponds to the identity map $a \mapsto a$. The maximum (denoted by 1 or 1_p) corresponds to $a \mapsto \hat{p}$.

4.4.1 Interpretation and examples.

The + operator yields the most desirable system incorporating the effect of both of its operands. The $\cdot$ operator dually yields the least desirable system whose effects are contained within both of its operands. Their use and significance is partially illustrated through the following six examples.

Example 0. Intuitive interpretation of the + operator

The + operation combines the rules of the systems. If each of f and g is seen to be described by a set of monotone deduction rules, then $f + g$ is the system that is obtained from the union of these sets of rules. The intuitive picture of combining rules may also found in the characterization $f + g = (fg)^{|P|}$. Both rules of f and g are iteratively applied on an initial state to yield a final state. Furthermore, the order of composition does not affect the final state, as long as each system is applied enough times. This insight follows from the interaction of A.1 and A.2, and is made formal in Subsection 4.3.5.

In a societal setting, each agent' state is governed by a set of local rules. Every such set only affects the state of its corresponding agent. The aggregate (via +) of all the local rules then defines the whole system. It allows for an interaction between the rules, and makes way for cascade effects to emerge. In the context of failures in infrastructure, the + operator enables adding new conditions for failure/disconnections in the system. This direction of aggregating local rules is further pursued in Section 4.5 on component realization. The definition of cascade effects is further expounded in Section 4.7. The five examples to follow also provide additional insight.

Example 1. Overview on M.0

Let f and f' be systems derived from instances (V, A) and (V, A') of M.0. If $A' \subseteq A$, then $f' \leq f$. If A' and A are non-comparable, an inequality may still hold as different digraphs may give rise to the same system. The system $f + f'$ is the system derived from $(V, A \cup A')$. The system $f \cdot f'$ is, however, not necessarily derived from $(V, A \cap A')$. If (V, A) is a directed cycle and (V, A') is the same cycle with the arcs reversed, then $f = f'$ while $(V, A \cap A')$ is the empty graph and yields the 0 system.

Example 2. Combining update rules

Given a set S, consider a subset $N_i \subseteq S$ and an integer k_i for each $i \in S$. Construct a map f_i that maps X to $X \cup \{i\}$ if $|X \cap N_i| \geq k_i$ and to X otherwise. Finally, define the map $f = f_1 + \cdots + f_n$. The map f can be realized by an instance of M.1, and each of the f_i corresponds to a *local evolution rule*.

Example 3. Recovering update rules

Given the setting of the previous example, define the map $e_i : X \mapsto X \cup \{i\}$. This map enables the extraction of a *local* evolution rule. Indeed, $i \in (f \cdot e_i)X_0$ if, and only if, $i \in fX_0$. However, if $j \neq i$, then $j \in (f \cdot e_i)X_0$ if, and only if, $j \in X_0$. It will later be proved that $f = f \cdot e_1 + \cdots + f \cdot e_n$. The system f can be realized as a combination of evolution rules, each governing the behavior of only one element of S.

Example 4. An instance of Boolean systems

Consider the following two instances of M.4, where L is the Boolean lattice. Iteration indices are dropped in the notation.

$$\begin{aligned} x_1 &:= x_1 \vee (x_2 \wedge x_3) & \qquad x_1 &= x_1 \\ x_2 &:= x_2 \vee x_3 & \qquad x_2 &= x_2 \vee x_3 \\ x_3 &:= x_3 & \qquad x_3 &= x_3 \vee (x_1 \wedge x_2) \end{aligned}$$

Let f and g denote the system maps generated by the right and left instances. The maps $f + g$ (left) and $f \cdot g$ (right) can then be realized as:

$$\begin{aligned} x_1 &:= x_1 \vee (x_2 \wedge x_3) & \qquad x_1 &= x_1 \\ x_2 &:= x_2 \vee x_3 & \qquad x_2 &= x_2 \vee x_3 \\ x_3 &:= x_3 \vee (x_1 \wedge x_2) & \qquad x_3 &= x_3 \end{aligned}$$

The map $f \cdot g$ is the identity map.

Example 5. Closure under Meet and Join.

If f and g are derived from instances of M.1, then neither $f+g$ nor $f \cdot g$ are guaranteed to be realizable as instances of M.1. If they are derived from instances of M.2, then only $f + g$ is necessarily realizable as an instance of M.2. As all systems (over the Boolean lattice) can be realized as instances of M.3, both $f + g$ and $f \cdot g$ can always be realized as instances of M.3.

As an example, we consider the case of M.2. If $(\mathcal{C}_f, k_f)$ and $(\mathcal{C}_g, k_g)$ are realizations of f and g as M.2, then $(\mathcal{C}_f \cup \mathcal{C}_g, k)$ is a realization of $f + g$, with k being k_f on $\mathcal{C}_f$ and k_g on $\mathcal{C}_g$. However, let $S = \{a, b, c\}$ be a set, and consider $\mathcal{C}_f = \{\{a, b\}\}$ with $k_f = 1$, and $\mathcal{C}_g = \{\{b, c\}\}$ with $k_g = 1$. The set $\{a, c\}$ is not a fixed-point of $f \cdot g$. Thus, if a realization $(\mathcal{C}_{f \cdot g}, k)$ of $f \cdot g$ is possible, then $\{a, b, c\} \in \mathcal{C}_{f \cdot g}$ with $k \leq 2$. However, both $\{a, b\}$ and $\{b, c\}$ are fixed-points of $f \cdot g$, contradicting such a realization.

4.4.2 Effect of the operators on fixed points.

The fixed point characterization uncovered thus far is independent of the order on $\mathcal{L}_P$. The map $\Phi : f \mapsto \{a : fa = a\}$ is also well behaved with respect to the $+$ and $\cdot$ operations. For $S, T \subseteq P$, we define their set meet $S \wedge T$ to be $\{a \wedge b : a \in S \text{ and } b \in T\}$.

Proposition 4.4.4. *If $f, g \in \mathcal{L}$, then $\Phi(f+g) = \Phi f \cap \Phi g$ and $\Phi(f \cdot g) = \Phi f \wedge \Phi g$.*

Proof. If $a \in \Phi f \cap \Phi g$, then $ga \in \Phi f$. As $fga = a$, it follows that $a \in \Phi(f+g)$. Conversely, as $(f+g)g = (f+g)$, if $(f+g)a = a$, then $(f+g)ga = a$ and so $ga = a$. By symmetry, if $(f+g)a = a$, then $fa = a$. Thus if $a \in \Phi(f+g)$, then $a \in \Phi f \cap \Phi g$. Furthermore, $(f \cdot g)a = a$ if, and only if, $fa \wedge ga = a$ and the result $\Phi(f \cdot g) = \Phi f \wedge \Phi g$ follows. □

Combination and projection lend themselves to simple operations when the maps are viewed as a collection of fixed points. Working directly in $\Phi\mathcal{L}$ will yield a remarkable conceptual simplification.

4.4.3 Summary on fixed points: the isomorphism theorem.

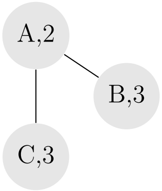
Let $\mathcal{F}$ be the collection of all $S \subseteq P$ such that $\hat{p} \in S$ and $a \wedge b \in S$ if $a, b \in S$. Ordering $\mathcal{F}$ by reverse inclusion $\supseteq$ equips it with a lattice structure. The join and meet of S and T in $\mathcal{F}$ are, respectively, set intersection $S \cap T$ and set meet $S \wedge T = \{a \wedge b : a \in S \text{ and } b \in T\}$. The set $S \wedge T$ may also be obtained by taking the union of S and T and closing the set under $\wedge$.

Theorem 4.4.5. *The map $\Phi : f \mapsto \{a : fa = a\}$ defines an isomorphism between $(\mathcal{L}, \leq, +, \cdot)$ and $(\mathcal{F}, \supseteq, \cap, \wedge)$.* □

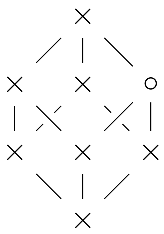
Such a result is well known in the study of *closure operators*, and is relatively simple. We refer the reader, for instance, to [Bir36], [Ore43] and [War42] for pieces of this theorem, and to [Bir67] Ch V and [CM03] for a broader overview, more insight and references. Nevertheless, the implications of it on the theory at hand can be remarkable. Our systems will be interchangeably used as both maps and subsets of P. The isomorphism enables a conceptual simplification, that enables emerging objects to be interpreted as systems exhibiting *cascade effects*.

4.4.4 Overview through an example (continued).

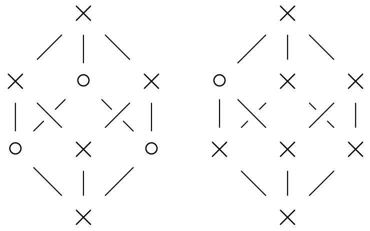
We continue the running example. Our example is realized as a *combination* of three evolution rules: one pertaining to each node. For instance, the rule of node A may be realized as:



The threshold 3 is just a large enough integer so that the colors of node B or node C do not change/evolve, regardless of the coloring on the graph. The system derived from such a realization is the map $f_A : 2^{\mathbf{3}} \to 2^{\mathbf{3}}$ satisfying A.1, A.2 and A.3 such that $BC \mapsto ABC$ and all remaining states are left unchanged. A fixed point representation yields:



Similarly the maps f_B and f_C derived for the rules of B and C are represented (respectively from left to right) through their fixed points as:



Our overall descriptive rule of the dynamics is constructed by a *descriptive* combination of the evolution rules of A, B and C. With respect to the objects *behind* those rules, the overall system is obtained by a + combination of the *local* systems. Indeed we have $f = f_A + f_B + f_C$, and such a combination is obtained by only keeping the fixed points that are common to all three systems.

4.5 Components realization.

The systems derived from instances of "models" *forget* all the componental structure described by the model. Nodes in M.0 and M.1 are bundled together to form the Boolean lattice, and the system is a monolithic map from 2^V to 2^V. We have not discussed any means to recover components and interconnection structures from systems. We might want such a recovery for at least two reasons. First, we may be interested in understanding specific subparts of the modeled system. Second, we may want to realize our systems as instances of other models. In state spaces isomorphic to 2^S for some S, components may often be identified with the elements of S. In the case of M.0 and M.1, the components are represented as nodes in a graph. Yet, two elements of S might also be tightly coupled as to form a single component. It is also less clear what the components can be in non-Boolean lattices as state spaces. We formalize such a flexibility by considering the set $\mathcal{E}$ of all maps $0_q \times 1_{q'}$ in $\mathcal{L}_Q \times \mathcal{L}_{Q'} \subseteq \mathcal{L}_{Q \times Q'}$ for $Q \times Q' = P$. The map $0_q \times 1_{q'}$ sends $(q, q') \in Q \times Q'$ to $(q, \hat{q}')$ where $\hat{q}'$ is the maximum element of Q'. Indeed, the system 0_q, being the identity, keeps q unchanged in Q. The system $1_{q'}$, being the maximum system, sends q' to the maximum element $\hat{q}'$ of Q'. We refer to the maps of $\mathcal{E}$ as *elementary* functions (or systems). A *component* *realization* of P is a collection of systems $e_A, \cdots, e_H$ in $\mathcal{E}$ where:

$$\begin{aligned} & e_A + \cdots + e_H = 1 \\ & e_I \cdot e_J = 0 \quad \text{for all } I \neq J \end{aligned}$$

For a different perspective, we consider a direct decomposition of P into lattices $A, \cdots, H$ such that $A \times \cdots \times H = P$. An element t of P can be written either as a tuple $(t_A, \cdots, t_H)$ or as a string $t_A \cdots t_H$. If $(t_A, \cdots, t_H)$ and $(t'_A, \cdots, t'_H)$ are elements of P, then:

$$(t_A, \cdots, t_H) \vee (t'_A, \cdots, t'_H) = (t_A \vee t'_A, \cdots, t_H \vee t'_H) \tag{4.1}$$

$$(t_A, \cdots, t_H) \wedge (t'_A, \cdots, t'_H) = (t_A \wedge t'_A, \cdots, t_H \wedge t'_H). \tag{4.2}$$

Indeed, the join (resp. meet) in the product lattice, is the product of the joins (resp. meets) in the factor lattices. Maps $e_A, \cdots, e_H$ can be defined as $e_I : ti \mapsto t\hat{i}$, that keeps t unchanged and maps i to the maximum element $\hat{i}$ of I. These maps belongs to $\mathcal{L}_P$, and together constitute a component realization as defined above. Conversely, each component realization gives rise to a direct decomposition of P.

Theorem 4.5.1. *Let $e_A, \cdots, e_H$ be a component realization of P. If $f \in \mathcal{L}_P$, then $f = f \cdot e_A + \cdots + f \cdot e_H$.*

Proof. It is immediate that $f \cdot e_A + \cdots + f \cdot e_H \leq f$. To show the other inequality, consider $t \notin \Phi f$. Then $t_I \neq (ft)_I$ for some I. Furthermore, if $t' \geq t$ with $t'_I = t_I$, then $t' \notin \Phi f$. Assume $t \in \Phi(f \cdot e_I)$, then $t = s \wedge r$ for some $s \in \Phi f$ and $r \in \Phi e_I$. It then follows that $r_I = \hat{i}$, the maximum element of I. Therefore $s_I = t_I$ and $s \geq t$ contradicting the fact that $s \in \Phi f$. □

The map $f \cdot e_I$ may evolve only the I-th *component* of the state space.

Proposition 4.5.2. *If $s \in P$ is written as ti, then $(f \cdot e_I)s = t(fs)_I$, where $(fs)_I$ is the projection of fs onto the component I.*

Proof. We have $(f \cdot e_I)s = fs \wedge e_I s = f(ti) \wedge t\hat{i} = t(fs)_I$. The last equality follows from Equation 4.2. □

It is also the evolution rule governing the state of component I as a function of the full system state.

Proposition 4.5.3. *Let $e_A, \cdots, e_H$ be a component realization of P. If $f \in \mathcal{L}_P$, then $fa = (f \cdot e_A)a \vee \cdots \vee (f \cdot e_H)a$ for every $a \in P$.*

Proof. It is immediate that $(f \cdot e_A)a \vee \cdots \vee (f \cdot e_H)a \leq fa$. The other inequality follows from combining Proposition 4.5.2 and Equation 4.1. □

Example 4.5.4. *Let f be the system derived from an instance (V, A) of M.0. We consider the maps $e_i : X \mapsto X \cup \{i\}$ for $i \in V$. The collection $\{e_i\}$ forms a* component realization *where e_i corresponds to node i in the graph. The system $f \cdot e_i$ may be identified with the* ancestors *of i, namely, nodes j where a directed path from j to i exists. A realization (in the form of M.0) of $f \cdot e_i$ then colors i black whenever any* *ancestor of it is black, leaving the color of all other nodes unchanged. Combining the maps $f \cdot e_i$ recovers the map f.*

Interconnection structures (e.g. digraphs as used in M.1) may be further derived by defining projection and inclusion maps accordingly and requiring the systems to satisfy some fixed-point conditions. Such structures can be interpreted as systems in $\mathcal{L}_{\mathcal{L}_P}$. They will not be considered in this chapter.

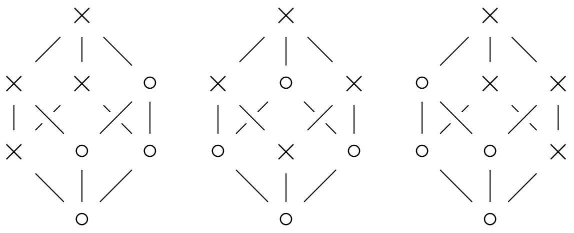
4.5.1 Defining cascade effects.

Given a component realization $e_A, \cdots, e_H$, define a collection of maps $f_A, \cdots, f_H$ where $f_I \leq e_I$ dictates the evolution of the state of component I as a function of P. These update rules are typically combined to form a system $f = f_A + \cdots + f_H$. *Cascade effects* are said to occur when $f \cdot e_I \neq f_I$ for some I. The behavior governing a certain (sub)system I is *enhanced* as this component is embedded into the greater system. We should consider the definition provided, in this subsection, as conceptually illustrative rather than useful and complete. The main goal of the chapter is to define a class of systems exhibiting cascade effects. It is not to define what cascade effects are. We instead refer the reader to [Ada17i] for an actionable definition and a study of these effects. We will however revisit this definition in Section 4.7 with more insight.

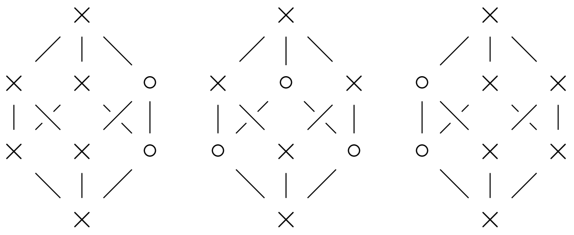
The conditions under which such effects occurs depend on the properties of the operations. If $\cdot$ distributes over $+$, then this behavior is never bound to occur; this will seldom be the case as will be shown in the next section.

4.5.2 Overview through an example (continued).

We continue the running example. On a dual end, if we wish to view the nodes A, B and C as distinct entities, we may define a component realization e_A, e_B and e_C represented (respectively from left to right) as:



Local evolution rules may be recovered through the systems $f \cdot e_A$, $f \cdot e_B$ and $f \cdot e_C$. Those are likely to be different than f_A, f_B and f_C as they also take into account the *effects* resulting from their combination. The systems $f \cdot e_A$, $f \cdot e_B$ and $f \cdot e_C$ are generated by considering $\Phi f \cup \Phi e_I$ and closing this set under $\cap$. They are represented (respectively from left to right) as:



The system $f \cdot e_A$ captures the fact that node A can become black if only C is colored black. A change in f_A would, however, require both B and C to be black. Recombining the obtained local rules is bound to recover the overall system, and indeed $f = f \cdot e_A + f \cdot e_B + f \cdot e_C$ as can be checked by keeping only the common fixed points.

4.6 Properties of the systems lattice.

Complex systems will be built out of *simpler* systems through expressions involving $+$ and $\cdot$. The power of such an expressiveness will come from the properties exhibited by the operators. Those are trivially derived from the properties of the lattice $\mathcal{L}$ itself.

Proposition 4.6.1. *The following propositions are equivalent. (i) The set P is linearly ordered. (ii) The lattice $\mathcal{L}_P$ is distributive. (iii) The lattice $\mathcal{L}_P$ is modular.*

Proof. Property (ii) implies (iii) by definition. If P is linearly ordered, then $\mathcal{L}$ is a Boolean lattice, as any subset of P is closed under $\wedge$. Therefore (i) implies (ii). Finally, it can be checked that (f, g) is a modular pair if, and only if, $\Phi(f \cdot g) = \Phi(f) \cup \Phi(g)$ i.e., $\Phi(f) \cup \Phi(g)$ is closed under $\wedge$. If $\mathcal{L}_P$ is modular, then each pair of f and g is modular. In that case, each pair of states in P are necessarily comparable, and so (iii) implies (i). □

The state spaces we are interested in are not linearly ordered. Non-distributivity is natural within the interpreted context of cascade effects, and has at least two implications. First, the decomposition of Theorem 4.5.1 cannot follow from distributivity, and relies on a more subtle point. Second, cascade effects (as defined in Section 4.5) are bound to occur in non-trivial cases.

The loss of modularity is suggested by the asymmetry in the behavior of the operator. The $+$ operator corresponds to set intersection, whereas the $\cdot$ operator (is less convenient) corresponds to a set union followed by a closure under $\wedge$. Nevertheless, the lattice will be *half* modular.

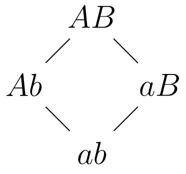
Proposition 4.6.2. *The lattice $\mathcal{L}_P$ is (upper) semimodular.*

Proof. It is enough to prove that if $f \cdot g \prec f$ and $f \cdot g \prec g$, then $f \prec f + g$ and $g \prec f + g$. If $f \cdot g$ is covered by f and g, then $|\Phi f - \Phi g| = |\Phi g - \Phi f| = 1$. Then necessarily $f + g$ covers f and g. □

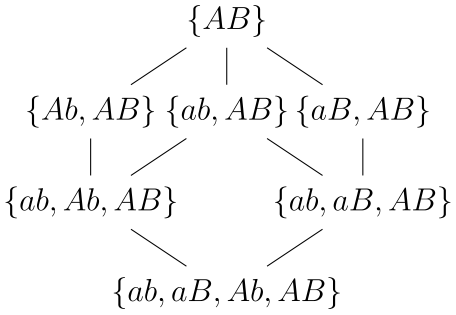
Semi-modularity will be fundamental in defining the μ-rank of a system in Section 7. The lattice $\mathcal{L}$ is equivalently a graded poset, and admits a rank function ρ such that $\rho(f+g)+\rho(f\cdot g)\leq\rho(f)+\rho(g)$. The quantity $\rho(f)$ is equal to the number of non-fixed points of f i.e. $|P-\Phi f|$. More properties may still be extracted, up to full characterization of the lattice. Yet, such properties are not needed in this chapter.

4.6.1 Additional remarks on the lattice of systems.

This subsection illustrates some basic lattice theoretic properties on $2^{\mathbf{2}}$, represented through its Hasse diagram below. We follow the notation of the running example (see e.g., Subsection 4.3.4).



The lattice $\mathcal{L}_{2^{\mathbf{2}}}$ may be represented as follows. The systems are labeled through their set of fixed-points.



A map $f\in\mathcal{L}_P$ will be called *prime* if $P-\Phi f$ is closed under $\wedge$. Those maps will be extensively used in Section 4.9.

All the systems are prime (i.e. have the set of non-fixed points closed under $\cap$) except for $\{ab,AB\}$. The lattice $\mathcal{L}_{2^{\mathbf{2}}}$ is (upper) semimodular as a pair of systems are covered by their join $(+)$ whenever they cover their meet $(\cdot)$. All pairs form modular pairs except for the pair $\{Ab,AB\}$ and $\{aB,AB\}$. The lattice $\mathcal{L}_{2^{\mathbf{2}}}$ is graded, and the (uniform) rank of a system is equal to the number of its non-fixed points as can be checked.

On atoms and join-irreducible elements.

An atom is an element that covers the minimal element of the lattice. In $\mathcal{L}_{2^{\mathbf{2}}}$, those are $\{ab,aB,AB\}$ and $\{ab,Ab,AB\}$. A join-irreducible element is an element that cannot be written as a join of *other* elements. An atom is necessarily a join-irreducible element, however the converse need not be true. The systems $\{aB,AB\}$ and $\{bA,AB\}$ are join-irreducible but are not atoms.

The join-irreducible elements in $\mathcal{L}_P$ may be identified with the pairs $(s,t)\in P\times P$ such that t covers s. They can be identified with the edges in the Hasse diagram of

P. For a covering pair (s, t), define f_{st} to be the least map such that $s \mapsto t$. Then f_{st} is join-irreducible for each (s, t), and every element of $\mathcal{L}_P$ is a join of elements in $\{f_{st}\}$.

Proposition 4.6.3. *The map f_{st} is prime for every (s, t).*

Proof. The map f_{st} is the least map such that $s \mapsto t$. It follows that s is the least non-fixed point of f_{st}, and that every element greater than t belongs to Φf_{st}. If $a, b \notin \Phi f_{st}$, then their meet $a \wedge b$ is necessarily not greater than t, for otherwise we get $a, b \in \Phi f_{st}$. If $a \wedge b$ is comparable to t, then $a \wedge b = s \notin \Phi f_{st}$. If $a \wedge b$ is non-comparable to t, then $(a \wedge b) \wedge t = s$, and so again $a \wedge b \notin \Phi f_{st}$. □

On coatoms and meet-irreducible elements.

A coatom is an element that is covered by the maximal element of the lattice. In $\mathcal{L}_{2^2}$, those are $\{ab, AB\}$, $\{aB, AB\}$ and $\{Ab, AB\}$. In general, the coatoms of $\mathcal{L}$ are exactly the systems f where $|\Phi f| = 2$. Note that the maximal element $\hat{p}$ of P is always contained in Φf.

Proposition 4.6.4. *Every $f \in \mathcal{L}_P$ is a meet of coatoms.*

Proof. For each $a \in P$, let $c_a \in \mathcal{L}$ be such that $\Phi c_a = \{a, \hat{p}\}$. If $\Phi f = \{a, b, \cdots, h\}$, then $f = c_a \cdot c_b \cdot \cdots \cdot c_h$. □

Such lattices are called co-atomistic. The coatoms, in this case, are the only elements that cannot be written as a meet of *other* elements.

4.7 On least fixed-points and cascade effects.

The systems are defined as maps $P \to P$ taking in an input and yielding an output. The interaction of those systems (via the operator $+$) however does not depend on functional composition or application. It is only motivated by them, and the input-ouput functional structure has been discarded throughout the analysis. It will then also be more insightful to not view $f(a)$ as functional application. Such a change of viewpoint can be achieved via a good use of least fixed-points. The change of view will also lead us the a more general notion of cascade effects.

We may associate to every $a \in P$ a system $\mathrm{free}(a) : - \mapsto - \vee a$ in $\mathcal{L}_P$. We can then interpret $f(a)$ differently:

Proposition 4.7.1. *The element $f(a)$ is the least fixed-point of $f + \mathrm{free}(a)$.*

Proof. We have $f(a) = \wedge\{p \in \Phi(f) : a \leq p\} = \wedge\{p \in \Phi(f) \cap \Phi(\mathrm{free}(a))\}$. The result follows as $\Phi(f) \cap \Phi(\mathrm{free}(a)) = \Phi(f + \mathrm{free}(a))$. □

The map $\mathrm{free} : P \to \mathcal{L}_P$ is order-preserving. It also preserves joins. Indeed, if $a, b \in P$, then $\mathrm{free}(a) + \mathrm{free}(b) = \mathrm{free}(a \vee b)$. Conversely, as each map in $\mathcal{L}_P$ admits a least fixed-point, we define $\mathrm{eval} : \mathcal{L}_P \to P$ to be the map sending a system to its least fixed-point. The map eval is also order-preserving, and we obtain:

Theorem 4.7.2. *If $a \in P$ and $f \in \mathcal{L}_P$, then:*

$$\text{free}(a) \leq f \quad \textit{if, and only if,} \quad a \leq \text{eval}(f)$$

Proof. If $\text{free}(a) \leq f$, then $a \leq b$ for every fixed-point b of f. Conversely, if $a \leq \text{eval}(f)$, then $\{b \in P : a \leq b\}$ contains $\Phi(f)$, the set of fixed points of f. □

The pair of maps free and eval are said to be adjoints, and form a Galois connection (see e.g., [Bir67] Ch. V, [Eve44], [Ore44] and [EKMS93] for a treatment on Galois connections). The intuition of cascading phenomena can be seen to partly emerge from this Galois connection. By duality, the map eval preserves meets. Indeed, the least fixed-point of $f \cdot g$ is the meet of the least fixed-points of f and g. The map eval does not however always preserve joins. Such a fact causes cascading intuition to arise. For some pairs $f, g \in \mathcal{L}_P$, we get:

$$\text{eval}(f+g) \neq \text{eval}(f) \vee \text{eval}(g) \tag{4.3}$$

Generally, two systems interact to yield, combined, something greater than what they yield separately, then combined. Specifically, consider $f \in \mathcal{L}_P$ and $a \in P$ such that $\text{eval}(f) \leq a$. If $\text{eval}(f + \text{free}(a)) \neq \text{eval}(f) \vee \text{eval}(\text{free}(a))$, then $f(a) \neq a$. In this case, the point a *expanded* under the map f, and cascading effects have thus occured. The chapter will not pursue this line of direction. This direction is extensively pursued in [Ada17i]. Also, a definition of cascade effects was already introduced in Section 4.5. We thus briefly revisit it and explain the connection to the inequality presented. The inequality can be further explained by the semimodularity of the lattice, but such a link will not be pursued.

4.7.1 Revisiting component realization.

Given a component realization $e_A, \cdots, e_H$ of P, we let $f_A, \cdots, f_H$ be a collection of maps where $f_I \leq e_I$ dictates the evolution of the state of component I as a function of P. If $f = f_A + \cdots + f_H$, then recall from Section 4.5 that *cascade effects* are said to occur when $f \cdot e_I \neq f_I$ for some I.

We will illustrate how this definition links to the inequality obtained from the Galois connection. For simplicity, we consider only two components A and B. Let e_A, e_B be a component realization of P, and consider two maps f_A, f_B where $f_I \leq e_I$. Define $f = f_A + f_B$. If $f \cdot e_A \neq f_A$, then $(f \cdot e_A)a \neq f_A a = a$ for some fixed point a of f_A. We then have $fa \neq f_A a \vee f_B a$. As $fa = (f \cdot e_A)a \vee (f \cdot e_B)a$ by Proposition 4.5.3, we get:

$$\text{eval}\big(f_A + \text{free}(a) + f_B + \text{free}(a)\big) \neq \text{eval}\big(f_A + \text{free}(a)\big) \vee \text{eval}\big(f_B + \text{free}(a)\big) \tag{4.4}$$

Conversely, if Equation 4.4 holds, then either $f_A a \neq (f \cdot e_A)a$ or $f_B a \neq (f \cdot e_B)a$.

4.7.2 More on Galois connections.

The inequality in Equation 4.3 gives rise to cascading phenomena in our situation. It is induced by the Galois connection between free and eval, and the fact that eval does not preserve joins. The content of the lattices can however be changed, keeping the phenomenon intact. Both the lattice of systems $\mathcal{L}_P$ and the lattice of states P can be replaced by other lattices. If we can setup another such inequality for the other lattices, then we would have created cascade effects in a different situation. We refer the reader to [Ada17i] for a thorough study along those lines. The particular class of systems studied in this chapter is however somewhat special. Indeed, every system itself arises from a Galois connection. Thus, if we focus on a particular system f, then we get a Galois connection induced by the inclusion:

$$\Phi(f) \to P$$

And indeed, cascade effects will emerge whenever $a \vee_{\Phi(f)} b \neq a \vee_P b$. This direction will not be further discussed in the chapter.

This double presence of Galois connection seems to be meerly a coincidence. It implies however that we can recover cascading phenomena in our situation at two levels: either at the level of systems interacting or at the level of a unique system with its states interacting.

4.7.3 Higher-order systems.

For a lattice P, we constructed the lattice $\mathcal{L}_P$. By iterating the construction once, we may form $\mathcal{L}_{\mathcal{L}_P}$. Through several iterations, we may recursively form $\mathcal{L}_P^{m+1} = \mathcal{L}_{\mathcal{L}_P^m}$ with $\mathcal{L}_P^0 = P$. Systems in $\mathcal{L}_P^m$ take into account nested if-then statements. The construction induces a map eval : $\mathcal{L}_P^{m+1} \to \mathcal{L}_P^m$, sending a system to its least fixed-point. We thus recover a sequence:

$$\cdots \to \mathcal{L}_P^3 \to \mathcal{L}_P^2 \to \mathcal{L}_P \to P$$

The free map construction induces an inclusion $\mathcal{L}_P^m \to \mathcal{L}_P^{m+1}$ for every m. We may then define an infinite lattice $\mathcal{L}_P^\infty = \bigcup_{m=1}^\infty \mathcal{L}_P^m$ that contains all finite higher-order systems. We may also decide to complete $\mathcal{L}_P^\infty$ in a certain sense to take into account infinite recursion. Such an idea have extensively recurred in denotational semantics and domain theory (see e.g., [Sco71], [SS71] and [Sco72]) to yield semantics to programming languages, notably the λ-calculus. This idea will however not be further pursued in this chapter.

4.8 Connections to formal methods.

The ideas developed in this chapter intersect with ideas in formal methods and semantics of languages. To clarify some intersections, we revisit the axioms. A map $f : P \to P$ belongs to $\mathcal{L}_P$ if it satisfies:

A.1 If $a \in P$, then $a \leq fa$.

A.2 If $a, b \in P$ and $a \leq b$, then $fa \leq fb$.

A.3 If $a \in P$, then $ffa = fa$.

The axiom A.2 may generally be replaced by one requiring the map to be *scott-continuous*, see e.g. [Sco72] for a definition. Every scott-continuous function is order-preserving, and in the case of finite lattices (as assumed in this chapter) the converse is true. The axiom A.3 may then be discarded, and fixed points can generally be recovered by successive iterations of the map (ref. the Kleene fixed-point theorem). The axiom A.1 equips the systems with their expansive nature. The more important axiom is A.2 (or potentially scott-continuity) which is adaptive to the underlying order. Every map satisfying A.2 can be *closed* into a map satisfying A.1 and A.2, by sending $f(-)$ to $- \vee f(-)$. The least fixed-points of both coincide.

The interplay of A.1 and A.2 ensures that concurrency of update rules in the systems does not produce any conflicts. The argument is illustrated in Proposition 4.4.3, and is further fully refined in Subsection 4.3.5. The systems can however capture concurrency issues by considering power sets. As an example, given a Petri net, we may construct a map sending a set of initial token distribution, to the set of all possible token distributions that can be *caused* by such an initial set. This map is easily shown to satisfy the axioms A.1, A.2 and A.3. A more elaborate interpretation of the state space, potentially along the lines of event structures as described in [NPW81], may lead to further connections for dealing with concurrency issues.

The interplay of lattices and least fixed-point appears throughout efforts in formal methods and semantics of languages. We illustrate the relevance of A.1 and A.2 via the simple two-line program `Prog`:

```
1. while ( x > 5 ) do
2. x := x - 1;
```

We define a state of this program to be an element of $\Sigma := \mathbb{N} \times \{in_1, out_1, in_2, out_2\}$. A number in $\mathbb{N}$ denotes the value assigned to `x`, and in_i (resp. out_i) indicates that the program is entering (resp. exiting) line i of the program. For instance, $(7, out_2)$ denotes the state where `x` has value 7 right after executing line 2. We define a finite execution trace of a program to be a sequence of states that can be reached by some execution of the program in finite steps. A finite execution trace is then an element of Σ^*, the semigroup of all finite strings over the alphabet Σ. Two elements $s, s' \in \Sigma^*$ can be concatenated via $s \circ s'$. We then define $f : 2^{\Sigma^*} \to 2^{\Sigma^*}$ such that:

$$\begin{aligned} B \mapsto f(B) :=& \big\{(n, in_1) : n \in \mathbb{N}\big\} \\ & \cup \big\{tr \circ (n, out_1) : tr \in B \text{ and } tr \in \Sigma^* \circ (n, in_1)\big\} \\ & \cup \big\{tr \circ (n, in_2) : tr \in B \text{ and } tr \in \Sigma^* \circ (n, out_1) \text{ and } n > 5\big\} \\ & \cup \big\{tr \circ (n, out_2) : tr \in B \text{ and } tr \in \Sigma^* \circ (n+1, in_2)\big\} \\ & \cup \big\{tr \circ (n, in_1) : tr \in B \text{ and } tr \in \Sigma^* \circ (n, out_2)\big\} \end{aligned} \tag{4.5}$$

The map f satisfies A.1 and A.2. If $B_{sol} \subseteq \Sigma^*$ is the set of finite excution traces, then $B_{sol} \supseteq f(B_{sol})$. Furthermore, B_{sol} is the least fixed of f. This idea is pervasive in obtaining semantics of programs. The maps f, in deriving semantics, are however typically only considered to be order-preserving (or Scott-continuous). The connection to using maps satisfying both A.1 and A.2 somewhat hinges on the fact that for every order-preserving map h, the least fixed-point of $h(-)$ and $- \vee h(-)$ coincide. The map f may also be closed under A.3 via successive iterations, without modifying the least fixed-point, to yield a map in $\mathcal{L}_{2^{\Sigma^*}}$. We refer the reader to [NNH15] Ch 1 for an overview of various methods along the example we provide, the work on abstract interpretation (see e.g., [CC77] and [Cou01]) for more details on traces and semantics, and the works [Sco71], [SS71] and [Sco72] for the relevance of A.2 (or Scott-continuity) in denotational semantics. In a general poset, non-necessarily boolean, we recover the form of M.4. Galois connections also appear extensively in abstract interpretation. The methods of abstract interpretation can be enhanced and put to use in approximating (and further understanding) the systems in this chapter.

Various ideas present in this chapter may be further linked to other areas. That ought not be surprising as the axioms are very minimal and natural. From this perspective, the goal of this work is partly to guide efforts, and very effective tools, in the formal methods community into dealing with cascade-like phenomena.

4.8.1 Cascading phenomena in this context.

We also illustrate cascade effects, as described in Section 4.7, in the context of programs. Consider another program `Prog'`:

```
while ( x is odd ) do
x := x - 1;
```

Each of `Prog` and `Prog'` ought to be thought of as a partial description of a *larger* program. Their interaction yields the simplest program allowing both descriptions, namely:

```
while ( x > 5 ) or ( x is odd ) do
x := x - 1;
```

Let f and g be the maps (satisfying A.1 and A.2) attributed to `Prog` and `Prog'` respectively, as done along the lines of Equation 4.5. The set of finite execution traces of the combined program is then the least fixed-point of $f \vee g$, where $(f \vee g)B = fB \cup gB$. Note that $f \vee g$ then satisfies both A.1 and A.2. Cascade effects then appear upon interaction. The interaction of the program descriptions is bound to produce new traces that cannot be accounted for by the traces of the separate programs. Indeed, every trace containing:

$$(5, out_2) \circ (5, in_1) \circ (5, out_1) \circ (5, in_2)$$

allowed in the combined program is not allowed in neither of the separate programs. Formally, define a map eval that sends a function $2^{\Sigma^*} \to 2^{\Sigma^*}$ satisfying A.1 and A.2

to its least fixed point. The map eval is well defined as 2^{Σ^*} is a complete lattice. We then get an inequality:

$$\text{eval}(f \vee g) \neq \text{eval}(f) \cup \text{eval}(g)$$

We may also link back to systems in $\mathcal{L}$ and the cascade effects' definition provided for them. If $\bar{f}$ and $\bar{g}$ denote the closure of f and g in 2^{Σ^*} to satisfy A.3 (e.g. via iterative composition in the case of scott-continuous functions), then the closure of $f \vee g$ corresponds to $\bar{f} + \bar{g}$. Of course, for every h satisfying A.1 and A.2, both h and $\bar{h}$ have the same least fixed-point. We then have:

$$\text{eval}(\bar{f} + \bar{g}) \neq \text{eval}(\bar{f}) \cup \text{eval}(\bar{g})$$

The chapter will mostly be concerned with properties of the systems in $\mathcal{L}$. The direction of directly studying the inequality will not be pursued in the chapter. It is extensively pursued in [Ada17i].

4.9 Shocks, failure and resilience.

The theory will be interpreted within cascading failure. The informal goal is to derive conditions and insight determining whether or not a system hit by a shock would fail. Such a statement requires at least three terms—*hit*, *shock* and *fail*—to be defined.

The situation, in the case of the models M.i, may be interpreted as follows. Some components (or agents) initially fail (or become infected). The dynamics then lead other components (or agents) to fail (or become infected) in turn. The goal is to assess the conditions under which a large fraction of the system's components fail. Such a state may be reached even when a very small number of components initially fail. This section aims to quantify and understand the resilence of the system to initial failures. Not only may targeted componental failures be inflicted onto the system, but also external (exogenous) rules may act as shocks providing conditional failures in the systems. A shock in this respect is to be regarded as a system. This remark is the subject of the next subsection.

4.9.1 A notion of shock.

Enforcing a *shock* on a system would intuitively yield an evolved system incorporating the effects of the shock. Forcing such an intuition onto the identity system leads us to consider shocks as systems themselves. Any shock s is then an element of $\mathcal{L}_P$. Two types of shocks may further be considered. *Push shocks* evolve state $\check{p}$ to some state a. *Pull shocks* evolve some state a directly to $\hat{p}$. Allowing arbitrary $+$ and $\cdot$ combinations of such systems generates $\mathcal{L}$. The set of shocks is then considered to be the set $\mathcal{L}$.

Shocks trivially inherit all properties of systems, and can be identified with their fixed points as subsets of P. Finally, a shock s *hits* a system f to yield the system $f + s$.

Example 4.9.1. *One example of shocks (realized through the form of M.i) inserts element to the initial set X_0 to obtain X_0'. This shock corresponds to the (least) map in $\mathcal{L}$ that sends $\emptyset$ to X_0'. Equivalently, this shock has as a set of fixed points the principal (upper) order filter of the lattice P generated by the set X_0' (i.e. the fixed points are all, and only, the sets containing X_0'). Further shocks may be identified with decreasing k_i or adding an element j to N_i for some i.*

Remark

It will often be required to restrict the space of shocks. There is no particular reason to do so now, as any shock can be well justified, for instance, in the setting of M.3. We may further wish to keep the generality to preserve symmetry in the problem, just as we are not restricting the set of systems.

4.9.2 A notion of failure.

A shock is considered to fail a system if the mechanisms of the shock combined with those of the system evolve the most desirable state to the least desirable state. Shock s fails system f if, and only if, $s + f = 1$.

In the context of M.i, failure occurs when $X_{|S|}$ contains all the elements of S. This notion of failure is not restrictive as it can simulate other notions. As an example, for $C \subseteq P$, define $u_C \in \mathcal{L}$ to be the least system that maps a to $\hat{p}$ if $a \in C$. Suppose shock s "fails" f if $(f+s)a \geq c$ for some $c \in C$ and all a. Then s "fails" f if, and only if, $f + s + u_c = 1$. The notion may further simulate notions of failure arising from monotone propositional sentences. If we suppose that (s_1, s_2, s_3) "fails" (f_1, f_2, f_3) if (s_1 fails f_1) and (either s_2 fails f_2 or s_3 fails f_3), then there is a map ψ into $\mathcal{L}$ such that (s_1, s_2, s_3) "fails" (f_1, f_2, f_3) if, and only if, $\psi(s_1, s_2, s_3) + \psi(f_1, f_2, f_3) = 1$. We can generally construct a monomorphism $\psi : \mathcal{L}_P \times \mathcal{L}_Q \to \mathcal{L}_{P\times Q}$ such that $s + f = 1$ and (or) $t + g = 1$ if, and only if, $\psi(s,t) + \psi(f,g) = 1$.

4.9.3 Minimal shocks and weaknesses of systems.

We set to understand the class of shocks that fail a system. We define the collection $\mathcal{S}_f$:

$$\mathcal{S}_f = \{s \in \mathcal{L} : f + s = 1\}$$

As a direct consequence of Theorem 4.4.5, we get:

Corollary 4.9.2. *Shock s belongs to S_f if, and only if, $\Phi f \cap \Phi s = \{\hat{p}\}$*

For instances of M.i, it is often a question as to whether or not there is some X_0 with at most k elements, where the final set $X_{|S|}$ contains all the elements of S. Such a set exists if, and only if, for some set X of size k, all sets containing it are non-fixed points (with the exception of S).

If $s \leq s'$ and $s \in S_f$, then $s' \in S_f$. Thus, an understanding of $\mathcal{S}_f$ may come from an understanding of its minimal elements. We then focus on the *minimal shocks* that fail a system f, and denote the set of those shocks by $\check{S}_f$:

$$\check{\mathcal{S}}_f = \{s \in \mathcal{S}_f : \text{for all } t \in S_f, \text{ if } t \leq s \text{ then } t = s\}$$

A map $f \in \mathcal{L}_P$ will be called *prime* if $P - \Phi f$ is closed under $\wedge$. A prime map f is naturally complemented in the lattice, and we define $\neg f$ to be (the prime map) such that $\Phi(\neg f) = P - (\Phi f - \{\hat{p}\})$. If f is prime, then $\neg\neg f = f$.

Proposition 4.9.3. *The system f admits a unique minimal shock that fails it, i.e. $|\check{\mathcal{S}}_f| = 1$ if, and only if, f is prime.*

Proof. If f is prime, then $\neg f \in \mathcal{S}_f$. The map $\neg f$ is also the unique minimal shock as if $s \in \mathcal{S}_f$, then $\Phi s \subseteq \Phi\neg f$ by Proposition 4.9.2. Conversely, suppose f is not prime. Then $a = b \wedge c$ for some $a \in \Phi f$ and $b, c \notin \Phi f$. Define $b' = fb$ and $c' = fc$ and consider the least shocks $s_0, s_{b'}$ and $s_{c'}$ such that $s_0\check{p} = a, s_{b'}b' = \hat{p}$ and $s_{c'}c' = \hat{p}$. Furthermore, define s_b and s_c such that $s_b a = b$ and $s_c a = c$. Then $b \in \Phi s_b$ and $c \in \Phi s_c$. Finally, $s_0 + s_b + s_{b'}$ and $s_0 + s_c + s_{c'}$ belong to $\mathcal{S}_f$, but their meet is not in $\mathcal{S}_f$ as a is a fixed point of $(s_0 + s_b + s_{b'}) \cdot (s_0 + s_c + s_{c'})$. This contradicts the existence of a minimal element in $\mathcal{S}_f$. □

As an example, consider an instance of M.1 where "the underlying graph is undirected" i.e. $i \in N_j$ if, and only if, $j \in N_i$. Define f to be the map $X_0 \mapsto X_{|S|}$. If $f(\emptyset) = \emptyset$ and $f(S - \{i\}) = S$ for all i, then $|\check{S}_f| \neq 1$ i.e. there are at least two minimal shock that fail f. Indeed, consider a minimal set X such that $fX \neq X$. If $Y = (X \cup N_i) - \{i\}$ for some $i \in X$, then $fY \neq Y$. However, $f(X \cap Y) = X \cap Y$ by minimality of X.

Theorem 4.9.4. *If s belongs to $\check{\mathcal{S}}_f$, then s is prime.*

Proof. Suppose s is not prime. Then, there exists a minimal element $a = b \wedge c$ such that $a \in \Phi s$ and $b, c \notin \Phi s$. We consider (b, c) to be *minimal* in the sense that for $(b', c') \neq (b, c)$, if $b' \wedge c' = a$, $b' \leq b$ and $c' \leq c$ then either $b' \in \Phi s$ or $c' \in \Phi s$. As $a \in \Phi s$ and $s \in \mathcal{S}_f$, it follows that $a \notin \Phi f$. Therefore, at least one of b or c is not in Φf. Without loss of generality, suppose that $b \notin \Phi f$. If for each $x \in \Phi s$ non-comparable to b, we show that $b \wedge x \in \Phi s$, then it would follow that s is not minimal as $\Phi s \cup \{b\}$ is closed under $\wedge$ and would constitute a shock $s' \leq s$ that fails f. Consider $x \in \Phi s$, and suppose $b \wedge x \notin \Phi s$. If $a \leq x$, then we get $(b \wedge x) \wedge c = a$ contradicting the minimality of (b, c). If a and x are not comparable, then $a \wedge x \neq a$. But $a \wedge x \in \Phi s$ and $a \wedge x = (b \wedge x) \wedge c$ with both $(b \wedge x)$ and c not in Φs, contradicting the minimality of a. □

Dually, we define the set of prime systems *contained* in f.

$$\mathcal{W}_f = \{w \leq f : w \text{ is prime}\}$$

Proposition 4.9.5. *If $f \in \mathcal{L}$ and $\mathcal{W}_f = \{w_1, \cdots, w_m\}$, then $f = w_1 + \cdots + w_m$.*

Proof. All join-irreducible elements of $\mathcal{L}$ are prime (see Subsection 4.6.1). Therefore $\mathcal{W}_f$ contains all join-irreducible elements less than f, and f is necessarily the join of those elements. □

Keeping only the maximal elements of $\mathcal{W}_f$ is enough to reconstruct f. We define:

$$\hat{\mathcal{W}}_f = \{w \in \mathcal{W}_f : \text{for all } v \in \mathcal{W}_f,\ \text{if } w \leq v \text{ then } v = w\}$$

Proposition 4.9.6. *The operator $\neg$ maps $\check{\mathcal{S}}_f$ to $\hat{\mathcal{W}}_f$ bijectively.*

Proof. If f is prime, then $\neg\neg f = f$. It is therefore enough to show that if $s \in \check{\mathcal{S}}_f$, then $\neg s \in \hat{\mathcal{W}}_f$ and that if $w \in \hat{\mathcal{W}}_f$, then $\neg w \in \check{\mathcal{S}}_f$. For each $s \in \check{\mathcal{S}}_f$, as $\neg s \leq f$, there is a $w \in \hat{\mathcal{W}}_f$ such that $\neg s \leq w$. Then $\neg w \leq s$, and so $s = \neg w$ as s is minimal. By symmetry we get the result. □

We will term prime functions in $\mathcal{W}_f$ as *weaknesses* of f. Every system can be decomposed injectively into its maximal weaknesses, and to each of those weaknesses corresponds a unique minimal shock that leads a system to failure. A minimal shock fails a system because it complements one maximal weakness of the system. Furthermore, whenever an arbitrary shock s fails f that is because a prime subshock s' of s complements a weakness w in f.

4.9.4 μ-Rank, resilience and fragility.

We may wish to quantify the *resilience* of a system. One interpretation of it may be the minimal amount of *effort* required to fail a system. The word *effort* presupposes a mapping that assigns to each shock some magnitude (or energy). As shocks are systems, such a mapping should coincide with one on systems.

Let $\mathbb{R}^+$ denote the non-negative reals. We expect a notion of magnitude $r : \mathcal{L} \to \mathbb{R}^+$ on the systems to satisfy two properties.

R.1 $r(f) \leq r(g)$ if $f \leq g$

R.2 $r(f + g) = r(f) + r(g) - r(f \cdot g)$ if (f, g) are modular.

The less desirable a system is, the higher the magnitude the system has. It is helpful to informally think of a modular pair (f, g) as a pair of systems that do not *interfere* with each other. In such a setting, the magnitude of the combined system adds up those of the subsystems and removes that of the common part.

The rank function ρ of $\mathcal{L}$ necessarily satisfies R.1 and R.2 as $\mathcal{L}$ is semi-modular. It can also be checked that, for any additive map $\mu : 2^P \to \mathbb{R}^+$, the map $f \mapsto \mu(P - \Phi f)$ satisfies the two properties. Thus, measures μ on P can prove to be a useful source for maps capturing magnitude. However, any notion of magnitude satisfying R.1 and R.2 is necessarily induced by a measure on the state space.

Theorem 4.9.7. *Let r be a map satisfying R.1 and R.2, then there exists an additive map $\mu : 2^P \to \mathbb{R}^+$ such that $r(f) = \mu(P - \Phi f) + r(0)$.*

Proof. A co-atom in $\mathcal{L}$ is an element covered by the system 1. For each f, there is a sequence of co-atoms $c_1, \cdots, c_m \in \mathcal{L}$ such that if $f_i = c_1 \cdot \cdots \cdot c_i$, then (f_i, c_{i+1}) is a modular pair, $f_i + c_{i+1} = 1$ and $f_m = f$. It then follows by R.2 that $r(f_i + c_{i+1}) = r(f_i) + r(c_{i+1}) - r(f_i \cdot c_{i+1})$. Therefore $r(f) = r(1) - \sum_{i=1}^{m} r(1) - r(c_i)$. Let c_a be the co-atom with $a \in \Phi c_a$, and define $\mu(\{a\}) = r(1) - r(c_a)$ and $\mu(\{\hat{p}\}) = 0$. It follows that $r(0) = r(1) - \mu(P)$ and so $r(f) = r(0) + \mu(P) - \mu(\Phi f)$. Equivalently $r(f) = \mu(P - \Phi f) + r(0)$. □

As it is natural to provide the identity system 0 with a zero magnitude, we consider only maps r additionally satisfying:

R.3 $r(0) = 0$.

Let r be a map satisfying R.1, R.2 and R.3 induced by the measure μ. If $\mu S = |S|$, then r is simply the rank function ρ of $\mathcal{L}$. We thus term r (for a general μ) as a μ-rank on $\mathcal{L}$. The notion of μ-rank is *similar* to that of a norm as defined on Banach spaces. Scalar multiplication is not defined in this setting, and does not translate (directly) to the algebra presented here. However, the μ-rank does give rise to a metric on $\mathcal{L}$.

Example 4.9.8. *Let f be the system derived from an instance (V, A) in M.0, and let μ be the counting measure on 2^V i.e. $\mu S = |S|$. If A is symmetric, then the system f has 2^c fixed points where c is the number of connected components in the graph. The μ-rank of f is then $2^{|V|} - 2^c$.*

Let r be a μ-rank. The quantity we wish to understand (termed *resilience*) would be formalized as follows:

$$res(f) = \min_{s \in \mathcal{S}_f} r(s)$$

We may dually define the following notion (termed *fragility*):

$$fra(f) = \max_{w \in \mathcal{W}_f} r(w)$$

Proposition 4.9.9. *We have $fra(f) + res(f) = r(1)$*

Proof. We have $\min_{s \in \check{S}_f} r(s) = \min_{w \in \hat{W}_f} r(\neg w)$ and $r(\neg w) = r(1) - r(w)$ for $w \in \hat{W}_f$. □

Example 4.9.10. *Let f be the system derived from an instance (V, A) in M.0, and let μ be the counting measure on 2^V i.e. $\mu S = |S|$. If A is symmetric, then the resilience/fragility of f is tied to the size of the largest connected component of the graph. Let us define $n = |V|$. If (V, A) had one component, then $res(f) = 2^{n-1}$. If (V, A) had m components of sizes $c_1 \geq \cdots \geq c_m$, then $res(f) = 2^{n-1} + 2^{n-c_1-1} + \cdots + 2^{n-(c_1+\cdots+c_{m-1})-1}$. As $r(1) = 2^n - 1$, it follows that $fra(f) = 2^n - 1 - res(f)$.*

The quantity we wish to understand may be either one of res or fra. However, the dual definition fra puts the quantity of interest on a comparable ground with the μ-rank of a system. It is always the case that $fra(f) \leq r(f)$. Furthermore, equality is not met unless the system is prime. It becomes essential to quantify the inequality gap. Fragility arises only from a certain *alignment* of the non-fixed points of the systems, formalized through the *prime* property. Not all high ranked systems are fragile, and combining systems need not result in fragile systems although rank is increased. It is then a question as to whether or not it is possible to combine resilient systems to yield a fragile systems. To give insight into such a question, we note the following:

Proposition 4.9.11. *If $w \in \mathcal{W}_{f+g}$, then $w \leq u + v$ for some $u \in \mathcal{W}_f$ and $v \in \mathcal{W}_g$.*

Proof. As $\neg w + f + g = 1$, it follows that $f \in \mathcal{S}_{\neg w+g}$. Then there is a $u \leq f$ in $\check{\mathcal{S}}_{\neg w+g}$. As $\neg w + u + g = 1$, it follows that $g \in \mathcal{S}_{\neg w+u}$. Then is a $v \leq f$ in $\check{\mathcal{S}}_{\neg w+u}$. Finally, we have $\neg w + u + v = 1$, therefore $w \leq u + v$. □

Thus a weakness can only form when combining systems through a combination of weaknesses in the systems. The implication is as follows:

Corollary 4.9.12. *We have $fra(f + g) \leq fra(f) + fra(g)$.*

Proof. For every $w \in \mathcal{W}_{f+g}$, we have $r(w) \leq max_{(u,v)\in W_f \times \mathcal{W}_g} r(u) + r(v)$ as $w \leq u + v$ for some $u \in W_f$ and $v \in W_g$. □

It is not possible to combine two systems with low fragility and obtain a system with a significantly higher fragility. Furthermore, we are interested in the gap $r(f + g) - fra(f + g)$. If $fra(f) \geq fra(g)$, then $r(f) - 2fra(f)$ is a lower bound on the gap. One should be careful as such a lowerbound may be trivial in some cases. If P is linearly ordered, then $fra(f) = r(f)$ for all f. The bound in this case is negative. However, if P is a Boolean lattice and $\mu S = |S|$, then $r(f) - fra(f)$ may be in the order of $|P| = r(1)$ with $fra(f) \leq 2^{(-\log |P|)/2} r(f)$.

Other notions of resilience (eq. fragility) may be introduced. One notion can consider a convex combination of the μ-rank of the k highest-ranked shocks failing a system. The notion introduced in the chapter primarily serves to illustrate the type of insight our approach might yield. Any function on the minimal shocks (failing a system) is bound to translate to a dual function on weaknesses.

Remark

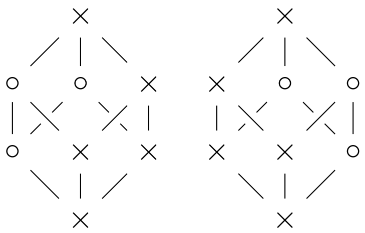
The statement of Corollary 4.9.12 may be perceived to be counterintuitive. This may be especially true in the context of cascading failure. The statement however should not be seen to indicate that the axioms defining a system and the dynamics preclude interesting phenomena. Indeed, it is the definition of fragility (and specifically the choice of the set of shocks over which we maximize) that gives rise to such a statement. The statement does not imply that fragility does not emerge from the combination of resilient systems, but only that we have a bound on how much fragility increases through combinations. The statement should not also diminish the validity of the definition of fragility, as it naturally arises from the mathematical structure of the problem. Another, potentially more intuitive, statement on *fragility* may however be recovered by a modification of the notion of fragility (or dually the notion of resilience) as follows.

We have considered so far every system to be a possible shock. Variations on the notion of resilience may be obtained by restricting the set of possible shocks. For instance, let us suppose that only systems of the form $s_a : p \mapsto p \vee a$ with $a \in P$ are possible shocks. In the case of boolean lattices, these shocks can be interpreted as initially marking a subset of components (or agents) as failed (or infected). These systems correspond, via their set of fixed-points, to the principal upper order filters of the lattice P. The notion of resilience then relates to the minimum number of initial failures (on the level of components) that lead to the failure of the whole system (i.e., all components). It is then rarely the case that two resilient systems when combined yield a resilient system. Indeed, if $a \vee b = \hat{p}$ with a and b distinct from $\hat{p}$, the maximum element of P, then both s_a and s_b have some resilence. The system $s_a + s_b$ has however no resilience at all, as it maps every p to the maximum element $\hat{p}$.

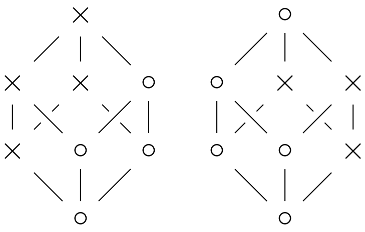
The space of possible shocks may be modified, changing the precise definition of fragility and yielding different statements. In case there are no restrictions on shocks, we obtain Corollary 4.9.12. We do not restrict shocks in the chapter, as a first analysis, due to the lack of a good reason to destroy symmetry between shocks and systems. The non-restriction allows us to capture the notion of a prime system and attain a characterization of fragility in terms of maximal weaknesses.

4.9.5 Overview through an example (continued).

We continue the running example. The maximal weaknesses of the system f are the maximal subsystems of f where the set of non-fixed points is closed under $\cap$. The system f has two maximal weaknesses, represented as:



The left (resp. right) weakness corresponds to the system failing when A (resp. C) is colored black. The left weakness is the map where $A \mapsto ABC$ leaving remaining states unchanged; the right weakness is the map where $C \mapsto ABC$ leaving remaining states unchanged. The system f then admits two corresponding minimal shocks that fail it. Those are complements to the weaknesses in the lattice.



The left (resp. right) minimal shock can be interpreted as initially coloring node A (resp. node C) black.

For a counting measure μ, the μ-rank of f is 5, whereas the fragility of f is 3. The resilience of f in that case is 4. For a system with non-trivial rules on the components, the lowest value of fragility attainable is 1. It is attained when all the nodes have a threshold of 2. The highest value attainable, however, is actually 3. Indeed, the system would have required the same amount of effort to fail it if all thresholds where equal to 1. Yet changing all the thresholds to 1 would necessarily increase the μ-rank to 6.

4.9.6 Recovery mechanisms and kernel operators.

Cascade effects, in this chapter, have been mainly driven by the axioms A.1 and A.2. The axiom A.1 ensures that the dynamics do not permit recovery. Those axioms however do not hinder us from considering situations where certain forms of recovery are permitted, e.g., when fault-protection mechanisms are built into the systems. Such situations may be achieved by dualizing A.1, and by considering multiple maps to define our fault-protected system. Specifically, we define a recovery mechanism k to be map $k : P \to P$ satisfying:

K.1 If $a \in P$, then $ka \leq a$.

A.2 If $a, b \in P$ and $a \leq b$, then $ka \leq kb$.

A.3 If $a \in P$, then $kka = ka$.

The axiom K.1 is derived from A.1 by only reversing the order. As such, a recovery mechanism k on P is only a system on the dual lattice P^{op}, obtained by reversing the partial order. The maps satisfying K.1, A.2 and A.3 are typically known as *kernel operators*, and inherit (by duality) all the properties of the systems described in this chapter.

We may then envision a system equipped with fault-protection mechanisms as a pair (k, f) where f is system in $\mathcal{L}_P$ and k is a recovery mechanism, i.e., a system in $\mathcal{L}_{P^{\mathrm{op}}}$. The pair (k, f) is then interpreted as follows. An initial state of failure is inflicted onto the system. Let $a \in P$ be the initial state. Recovery first occurs via the dynamics of k to yield a more desireable state $k(a)$. The *dynamics* of f then come into play to yield a state $f(k(a))$.

The collection of pairs (k, f) thus introduce a new class of systems, whose properties build on those developed in this chapter. If the axiom A.3 is discarded, iteration of maps in the form $(fk)^n$ may provide a more realistic account of the interplay of failures and recovery mechanisms. In general, the map fk will satisfy neither A.1 nor K.1. A different type of analysis might thus be involved to understand these new system.

Several questions may be posed in such a setting. For a design-question example, let us consider P to be a graded poset. What is the recovery mechanism k of minimum μ-rank, whereby $f(k(a))$ has rank (in P) less than l for every $a \in P$ with rank less than l'? Other design or analysis questions may posed, inspired by the example question. This direction of recovery however will not be further investigated in this chapter.

Remark

Another form of recovery may be achieved by *removing rules* from the system. Such a form may be acheived via the $\cdot$ operator. Indeed, the system $f \cdot g$ is the most undesireable system that includes the common rules of both f and g. If g is viewed as a certain complement of some system we want to remove from f, then we recover the required setting of recovery. The notion of complement systems is well-defined for prime systems. For systems that are not prime, it may be achieved by complementing the set of fixed-points, adding the maximum element $\hat{p}$ and then closing the obtained set under meets.

4.10 Concluding Remarks.

Finiteness is not necessary (as explained in Section 3) for the development. The axioms A.1, A.2 and A.3 can be satisfied when P is an infinite lattice, and Φf (for every f) is complete whenever P is complete. Nevertheless, the notion μ-rank should be *augmented* accordingly, and non-finite component realizations should be allowed. Furthermore, *semimodularity* on infinite lattices (still holds, yet) requires stronger conditions than what is presented in this chapter on finite lattices.

Finally, the choice of the state space and order relation allows a good flexibility in the modeling exercise. State spaces may be augmented accordingly to capture desired instances. But order-preserveness is intrinsic to what is developed. This said, hints of negation (at first sight) might prove not to be integrable in this framework.

Chapter 5

On the abstract structure of the behavioral approach to systems theory

Abstract

We revisit the behavioral approach to systems theory and make explicit the abstract pattern that governs it. Our end goal is to use that pattern to understand interaction-related phenomena that emerge when systems interact. Rather than thinking of a system as a pair $(\mathbb{U}, \mathcal{B})$, we begin by thinking of it as an injective map $\mathcal{B} \to \mathbb{U}$. This relative perspective naturally brings about the sought structure, which we summarize in three points. First, the separation of behavioral equations and behavior is developed through two spaces, one of syntax and another of semantics, linked by an interpretation map. Second, the notion of interconnection and variable sharing is shown to be a construction of the same nature as that of gluing topological spaces or taking amalgamated sums of algebraic objects. Third, the notion of interconnection instantiates to both the syntax space and the semantics space, and the interpretation map is shown to preserve the interconnection when going from syntax to semantics. This pattern, in its generality, is made precise by borrowing very basic constructs from the language of categories and functors.

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5.1 Introduction

Our primary concern is studying and understanding phenomena or behavior that arise from the interaction of several systems. We may describe the common situation of interest as small entities of systems coming together, interacting, and producing, as an aggregate, a behavior that would not have occurred without interaction. These situations are fundamental, and appear in countless settings, including contagion effects in societal systems and cascading failure in infrastructures. But a viable understanding of the emergent generated behavior ought to be preceded by an understanding of what it means to interconnect systems and have them interact. The paper begins as an attempt to grasp an understanding of interconnection.

The behavioral approach to systems theory, initiated by J. C. Willems in the 1980's proves to be a pedagogically responsible and natural approach to understanding interconnected systems. It provides, among other things, a sound inclusive definition of an open system, one that interacts with its environment, and develops interconnection through the natural notion of variable sharing. Rather than viewing a system as an input/output device, the behavioral approach views a system as a set—termed, *behavior*—of trajectories or outcomes deemed allowable by the laws of a mathematical model. Some of the theory's distinctive flair may be sketched through three points.

i. Behaviors are described by equations—termed, *behavioral equations*—and different equations may describe the same behavior. Intrinsic systemic properties then ought to be properties of the behavior, and not of the descriptive behavioral equations.

ii. Many systems are not fundamentally input/output devices, and as such signal-flow diagrams should not be natural as an interconnection construct. Instead, systems are interconnected through the notion of sharing variables.

iii. Interconnecting systems on the behavior level via variable sharing coincides with the descriptive interconnection of systems on the equational level. As Willems very often stressed (see e.g., [Wil07]): "thinking of a dynamical system as a behavior, and of interconnection as variable sharing, gets the physics right."

Our interest begins with the following questions. What is the abstract pattern that makes this theory so natural? How can we mathematically abstract away that pattern, simplify it, and use it in different settings?

Our approach, in answering those questions, consists of developing a relative point of view. Instead of thinking of a system as a pair $(\mathbb{U}, \mathcal{B})$ with $\mathcal{B} \subseteq \mathbb{U}$, as done in the behavioral approach, we first explicitly think of a system as an injective map $\mathcal{B} \to \mathbb{U}$ for arbitrary pairs of $\mathcal{B}$ and $\mathbb{U}$. This view leaves the definition of a system unchanged, but it forces us to introduce transformations, or morphisms, of systems, so as to relate the systems together. The analysis then proceeds naturally, and the findings are summarized again in three points, to reflect those raised above:

i. The separation of behavioral equations and behavior suggests a development of two *spaces*, one of syntactical objects (ref. the behavioral equations), and one of semantical objects (ref. the behavior) linked by an interpretation map.

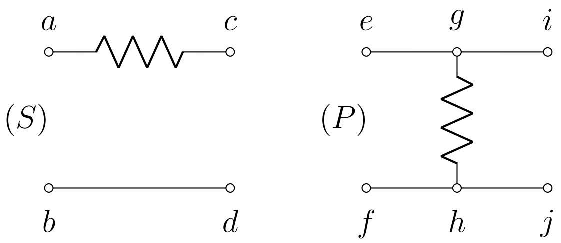
ii. The morphisms introduce a notion of subsystem and controlled-system. We show that interconnection, as variable sharing, amounts to *gluing* two systems on a common subsystem to yield a controlled-system. The intuition thus provided by the behavioral approach for interconnection through variable sharing lies on the same level as that of gluing topological spaces or taking amalgamated sums of algebraic objects.

iii. The notion of interconnection instantiates to both the syntax space and the semantics spaces, and the interpretation map is shown to preserve the interconnection when going from syntax to semantics.

The notion of interconnection in the behavioral approach is mathematically clear when the behaviors live in the same universum. It is less clear what that notion amounts to mathematically when the behaviors live in different universa. A recipe for interconnecting systems in different universa exists, but it does not directly lend itself to mathematical analysis. The key contribution lies in expressing the notion of interconnection through the notion of a pushout. In a linear setting (e.g., in linear systems), this will allow us to express interconnection of systems in terms of exact sequences from commutative algebra. Phenomena that emerge from the interaction of systems may then be seen to arise from a certain loss of exactness. We refer the reader to [Ada17a] for more details. The point (ii) further shows a duality between systems with latent variables on one end, and systems that are controlled on another. Altogether, the abstract notions exhibited by the behavioral approach may be instantiated to different settings. When those settings are interpreted as defining systems, we recover the intuition provided by the behavioral approach.

We begin by a brief review of the essential features in the behavioral approach, focusing particularly on three themes: the behavioral equations, variable sharing and latent variables. We then perform our shift of view to systems being injective maps, and revisit the three themes, highlighting the structure. The paper will introduce very basic elements of the functorial language (i.e., the language of categories and functors) along the way as needed. We finally end with a recap of the big picture, and a sketch of where the work leads to.

To dilute the abstraction, we will illustrate the claims by an example. The example runs throughout the paper, and consists, at various stages, of an interplay between two resistive circuits labelled (S) for series and (P) for parallel.



In functorial language, the behavioral approach suggests two categories, a syntax category (e.g., reflecting the behavioral equations) and a semantics category (e.g., reflecting the behavior). The categories are linked via an interpretation functor. Interconnection (e.g., through variable sharing) in both categories consists of taking pushouts or more generally colimits (or dually pullback/limits in the case of the behavioral approach). The interpretation functor preserves pushouts or more generally colimits, and may be desired to admit a right adjoint (e.g., reflecting that every behavior admits a *simplest* behavioral equation representation).

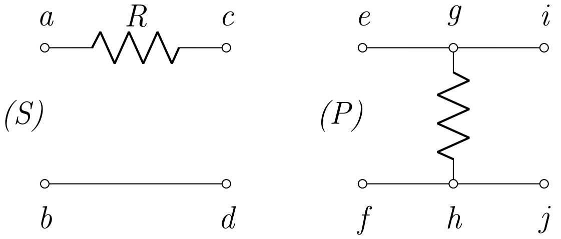
5.2 Jan Willems' behavioral approach.

The behavioral approach to systems theory begins from the premise that a mathematical model acts as an exclusion law. The phenomenon we wish to model produces events or outcomes that live in a given set $\mathbb{U}$. The laws of the model (viewed descriptively) state that some outcomes in $\mathbb{U}$ are possible, while others are not. The model then restricts the outcomes in $\mathbb{U}$ to only those that are allowed possible by the laws of the model. The set of possible outcomes is then called the *behavior* of a model. We refrain from using the term *model*, and replace it by *system*. Material in this section may be found in [PW98], [Wil91] and [Wil07].

Definition 5.2.1 (cf. [PW98], Section 1.2.1)**.** *A Willems system is a pair* $(\mathbb{U}, \mathcal{B})$ *where* $\mathbb{U}$ *is a set, called the universum—its elements are called* outcomes*—and* $\mathcal{B}$ *a subset of* $\mathbb{U}$ *called the* behavior*.*

The behavioral approach links naturally to standard ideas. A dynamical system may be obtained by considering universa of the form $\mathbb{W}^{\mathbb{T}}$, the set of maps from $\mathbb{T}$ to $\mathbb{W}$. The set $\mathbb{T}$ embodies the time axis, and $\mathbb{W}^{\mathbb{T}}$ then represents timed trajectories taking values in $\mathbb{W}$. An input-output structure can be recovered by thinking of a map as a relation. Every set map $f : A \to B$ defines a relation $R = \{(a, fa)\} \subseteq A \times B$ which yields a Willems system $(A \times B, R)$. The universa may also be endowed with additional structure, e.g., a vector space structure. A Willems system $(\mathbb{U}, \mathcal{B})$ may then be termed k-linear if $\mathbb{U}$ is a vector space over the field k, and $\mathcal{B}$ is a linear subspace of $\mathbb{U}$. Time invariance, among other things, for dynamical systems can be further brought into the picture. We refer the reader to [PW98], [Wil91] and [Wil07] for the details.

Example 5.2.2. *Considering the circuits (S) and (P),*



we declare the variables in play to be the voltage potentials $v_a, \cdots, v_j$, one for each labelled node, and currents $i_{ac}, i_{bd}, i_{eg}, i_{gi}, i_{gh}, i_{fh}, i_{hj}$, one for each consecutive pair of labelled nodes. We define $(\mathbb{U}_S, \mathcal{B}_S)$ and $(\mathbb{U}_P, \mathcal{B}_P)$ to be the Willems systems corresponding to (S) and (P). The universum $\mathbb{U}_S$ is the free $\mathbb{R}$-vector space (isomorphic to $\mathbb{R}^6$) generated by the basis $\{v_a, v_b, v_c, v_d, i_{ac}, i_{bd}\}$. The behavior $\mathcal{B}_S$ is the subset $\{(V_a, V_b, V_c, V_d, I_{ac}, I_{bd}) \in \mathbb{U}_S : V_a - V_c = RI_{ac}$ and $V_b = V_d\}$. Similarly, $\mathbb{U}_P$ is the $\mathbb{R}$-vector space (isomorphic to $\mathbb{R}^{11}$) generated by the variables that remain. The behavior $\mathcal{B}_P$ is the subset of $\mathbb{U}_P$ that satisfy KCL, KVL and Ohm's law.

5.2.1 Behavioral equations.

Systems may be generally described by equations. The behavior then consists of the outcomes for which *balance equations* are satisfied.

Definition 5.2.3 (cf. [PW98], Section 1.2.2)**.** *Let $\mathbb{U}$ be a universum, $\mathbb{E}$ a set, and $f_1, f_2 : \mathbb{U} \to \mathbb{E}$ maps. The Willems system $(\mathbb{U}, \mathcal{B})$ with $\mathcal{B} = \{u \in \mathbb{U} : f_1(u) = f_2(u)\}$ is said to be described by* behavioral equations *and is denoted by $(\mathbb{U}, \mathbb{E}, f_1, f_2)$. We call $(\mathbb{U}, \mathbb{E}, f_1, f_2)$ a* behavioral equation representation *of $(\mathbb{U}, \mathcal{B})$.*

If both $\mathbb{U}$ and $\mathbb{E}$ share a linear structure (e.g., are vector spaces), then the behavior of the representation $(\mathbb{U}, \mathbb{E}, f_1, f_2)$ is the kernel of $f_1 - f_2$. In such a setting, we talk about kernel representations of systems.

Systems are described in many situations using inequalities rather than equalities. Such a change may be remedied by considering $\mathbb{E}$ to be ordered. For instance, if $\mathbb{E}$ is a partially ordered set where every pair of elements (a, b) admit a least upper bound $max(a, b)$, then $f(u) \leq g(u)$ if, and only if, $max(f(u), g(u)) = g(u)$.

A system $(\mathbb{U}, \mathcal{B})$ may have different behavioral equation representations of it. It is then not the equations themselves that are essential, but rather the solution to those equations. This remark is the basis for a separation between syntax and semantics. The behavioral equations represent the syntax, while the semantics, the objects behind the syntax, are captured by the behavior.

Example 5.2.4. *The systems $(\mathbb{U}_S, \mathcal{B}_S)$ and $(\mathbb{U}_P, \mathcal{B}_P)$ possess a linear structure. Both $\mathcal{B}_S$ and $\mathcal{B}_p$ are the solution set of a system of linear equations. We can then explicitely define matrices (or linear maps) with the equations as rows, and obtain behavioral equation representations of the two systems.*

5.2.2 Interconnection and variable sharing.

The behavioral approach enables us to define interconnections of systems. Let $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ be Willems systems with representations $(\mathbb{U}, \mathbb{E}, f, g)$ and $(\mathbb{U}, \mathbb{E}', f', g')$, respectively. Their interconnection is the system represented by $(\mathbb{U}, \mathbb{E} \times \mathbb{E}', f \times f', g \times g')$.

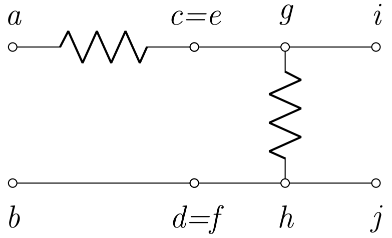
Definition 5.2.5. *The interconnection of* $(\mathbb{U}, \mathcal{B})$ *and* $(\mathbb{U}, \mathcal{B}')$ *is the system* $(\mathbb{U}, \mathcal{B} \cap \mathcal{B}')$. □

To interconnect two systems $(\mathbb{V} \times \mathbb{U}, \mathcal{B})$ and $(\mathbb{U} \times \mathbb{V}', \mathcal{B}')$ that share only a part $\mathbb{U}$ of their universa in common, we first lift them to two equivalent systems $(\mathbb{V} \times \mathbb{U} \times \mathbb{V}', \mathcal{B} \times \mathbb{V}')$ and $(\mathbb{V} \times \mathbb{U} \times \mathbb{V}', \mathbb{V} \times \mathcal{B}')$, and then intersect the lifted behaviors.

Definition 5.2.6. *The interconnection of* $(\mathbb{V} \times \mathbb{U}, \mathcal{B})$ *and* $(\mathbb{U} \times \mathbb{V}', \mathcal{B}')$ *by sharing* $\mathbb{U}$ *is the system* $(\mathbb{V} \times \mathbb{U} \times \mathbb{V}', \mathcal{B} \times \mathbb{V}' \cap \mathbb{V} \times \mathcal{B}')$. □

Variable sharing thus consists of declaring parts of the universa as representing the same outcomes, and carrying out the above procedure of identification. The identification is the basis for the *gluing* mentioned in the introduction. Definition 5.2.6 provides a mean to interconnect systems in different universa. However, that mean can be mathematically cumbersome. Part of the relative perspective to be developed goes into making interconnection less cumbersome when different universa are involved.

Example 5.2.7. *We will interconnect (S) and (P) by connecting terminal c to e, and d to f to obtain the circuit:*



To perform the interconnection, we need to identify the variables v_c*,* v_d*,* i_{ac} *and* i_{bd} *with* v_e*,* v_f*,* i_{eg} *and* i_{fh}*, respectively. The systems* $(\mathbb{U}_S, \mathcal{B}_S)$ *and* $(\mathbb{U}_P, \mathcal{B}_P)$ *need to be lifted to a common universum* $\mathbb{U}$*, in accordance with Definition 5.2.6, where every pair of the to-be-matched variables corresponds to the same dimension. The lifted behavior are then intersected.*

5.2.3 Latent variables.

The universum typically represents the variables that we wish to model. It is however often the case that auxiliary variables are needed. Adding auxiliary variables might lead to simpler behavioral equation representations. Interconnecting two systems that live in different universa will also force us to add auxiliary variables. Latent variables of auxiliary interest are then appended to the universum of the original manifest variables.

Definition 5.2.8 (cf. [PW98], Section 1.2.3)**.** *A Willems system with latent variables is defined as a triple $(\mathbb{U}, \mathbb{U}_l, \mathcal{B}_f)$ with $\mathbb{U}$ the universum of the manifest variables, $\mathbb{U}_l$ the universum of latent variables and $\mathcal{B}_f \subseteq \mathbb{U} \times \mathbb{U}_l$ the full behavior. It defines the manifest Willems system $(\mathbb{U}, \mathcal{B})$ with $\mathcal{B} := \{u \in \mathbb{U} : (u, l) \in \mathcal{B}_f \text{ for some } l \in \mathbb{U}_l\}$ where $\mathcal{B}$ is the manifest behavior. We call $(\mathbb{U}, U_l, \mathcal{B}_f)$ a latent variable representation of $(\mathbb{U}, \mathcal{B})$.*

Latent variables equip us with the extra flexibility needed in the modelling exercise. The theory of latent variables will thereafter appear in the notion of subsystems.

Example 5.2.9. *We may abstract the circuit (P) into a two-port blackbox, by declaring the universum to consist of the voltage potentials and currents at the four terminals e, f, i and j. The corresponding Willems system $(\mathbb{U}_{two-port}, \mathcal{B}_{two-port})$ consists of $\mathbb{R}^8$ as a universum, and the set of tuples in $\mathbb{R}^8$ that can physically coincide as the behavior. We define U_l to be $\mathbb{R}^3$ generated by the variables v_g, v_h and i_{gh}. The Willems system $(\mathbb{U}_P, \mathcal{B}_P)$ is equivalent to $(\mathbb{U}_{two-port}, \mathbb{U}_l, \mathcal{B}_P)$ and is then a latent variable representation of $(U_{two-port}, \mathcal{B}_{two-port})$.*

5.2.4 The immediate mathematical structure.

Let $\mathbb{U}$ be a fixed universum, and suppose that every subset of $\mathbb{U}$ is a potential behavior. We can partially order the behaviors in $\mathbb{U}$ by inclusion, and get a lattice $\mathcal{L}_{\mathbb{U}}$. The meet (min) in the lattice corresponds to set-intersection, and the join (max) corresponds to set-union. Interconnection of systems corresponds then to taking meets in the lattice. The properties of the lattice (as well as its existence) changes as different mathematical structures are imposed on the universa and the behaviors. For instance, if U is a vector space and the behaviors are the linear subspaces of $\mathbb{U}$, then the lattice of behaviors is modular. For a thorough study along those lines, we invite the reader to look at [Sha01].

5.3 The relative point of view.

We bring about the abstract structure by adopting a relative point of view. Instead of thinking of a system as a pair of sets $(\mathbb{U}, \mathcal{B})$ where $\mathcal{B} \subseteq \mathbb{U}$, we explicitly think of it as an injective map $\mathcal{B} \to \mathbb{U}$ of sets.

Remark: We only consider, in this section, universa and behaviors that are sets, without any additional structure. We can nevertheless equip the systems with more structure (such as an R-module structure) while keeping the insight and the result statements unchanged. We would however need to equip the set maps with a compatible structure. For instance, in the case of R-modules, the set maps would have to be replaced by R-linear maps.

5.3.1 Morphisms of systems.

A system is then an *injective* map $B \to U$ of sets. We will keep the labels B and U, instead of using other letters, simply to make the connection explicit with the behavioral approach as described. We will now revisit the above theory through the lens of injective maps. However, the systems thus far, as simply a collection of injective maps without any further structure, are unrelated. They cannot be interconnected and we cannot discuss most of the themes addressed in the previous section. We remedy this issue by defining a morphism of systems. The systems considered along with their morphisms will provide us with a *sandbox* to develop the theory we want.

Definition 5.3.1. *Let $s : B \to U$ and $s' : B' \to U'$ be two systems. A morphism $\boldsymbol{\phi}$ from s to s' denoted by $\boldsymbol{\phi} : s \Rightarrow s'$ is a pair of set maps (ϕ_B, ϕ_U) with $\phi_B : B \to B'$ and $\phi_U : U \to U'$ such that the diagram:*

$$\begin{tikzcd}
B \arrow[r, "s"] \arrow[d, "\phi_B"] & U \arrow[d, "\phi_U"] \\
B' \arrow[r, "s'"] & U'
\end{tikzcd}$$

commutes, i.e., such that $\phi_U s = s' \phi_B$.

If id_A denotes the identity map on the set A, then for every system $B \to U$, the pair (id_B, id_U) is a morphism of systems. Furthermore, morphism may be composed component-wise to yield other morphisms. Indeed, if $\boldsymbol{\phi} = (\phi_B, \phi_U) : s \Rightarrow s'$ and $\boldsymbol{\phi}' = (\phi'_B, \phi'_U) : s' \Rightarrow s''$ are morphisms, then the composition $\boldsymbol{\phi}'\boldsymbol{\phi} = (\phi'_B\phi_B, \phi'_U\phi_U) : s \Rightarrow s''$ is also a morphism.

Generally, given a diagram:

$$\begin{tikzcd}
B \arrow[r, "s"] \arrow[d, "\phi_B"] & U \arrow[d, "\phi_U"] \\
B' \arrow[r, "s'"] & U'
\end{tikzcd}$$

where s and s' are injective, it follows that ϕ_B is just the *restriction* of ϕ_U onto B. If either s or s' were not injective, then ϕ_B is not necessarily the restriction of ϕ_U. We cannot however always construct a commutative diagram by restricting an arbitrary ϕ_U onto B, as the image $\phi_U(B)$ may fall outside B'. We can then do so (if and) only if $\phi_U(B) \subseteq B'$.

The idea of a morphism, in a different form, appears in [Fuh01] and [Fuh02] through the notion of behavior homomorphism. Behavior homomorphisms were partly introduced to assist in settling problems regarding equivalence of system representations.

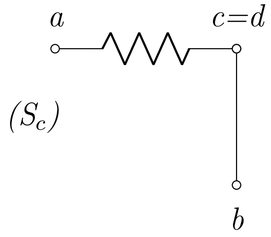
Subsystems and controlled-systems.

Introducing morphisms immediately introduces notions of a subsystem and a controlled-system.

Controlling the behavior of a system consists of restricting some of its potential outcomes. As such if $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ are Willems systems with $\mathcal{B} \subset \mathcal{B}'$, then $\mathcal{B}$ is a controlled version of $\mathcal{B}'$. Such a notion lifts naturally to the relative perspective.

Definition 5.3.2 (Controlled-system)**.** *Let $\boldsymbol{\phi} : s_{ctrl} \Rightarrow s$ be a morphism of systems. The pair $(s_{ctrl}, \boldsymbol{\phi})$ is said to be a controlled-system from s, if the components of $\boldsymbol{\phi}$ are injective maps. We may refer to s_{ctrl} as the controlled-system if $\boldsymbol{\phi}$ is clear from the context.*

Example 5.3.3. *The system underlying the circuit (S_c) can be seen as a controlled-system from that of (S):*

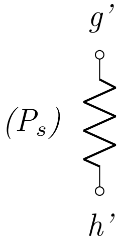


Let $s_c : \mathcal{B}_{S_c} \to \mathbb{U}_{S_c}$ and $s : \mathcal{B}_S \to \mathbb{U}_S$ be the systems of (S_c) and (S), respectively. Then $\mathbb{U}_{S_c}$ is the free $\mathbb{R}$-vector space with basis $\{v_{a'}, v_{b'}, v_{c'}, i_{a'c'}, i_{b'c'}\}$. The set $\mathcal{B}_{S_c}$ is the subset of $\mathbb{U}_{S_c}$ whose tuples satisfy the laws of the circuit. The morphism $\boldsymbol{\phi} : s_c \Rightarrow s$ is defined uniquely such that ϕ_U sends $v_{a'}, v_{b'}, v_{c'}, i_{a'c'}, i_{b'c'}$ in the basis of $\mathbb{U}_{S_c}$ respectively to $v_a, v_b, v_c + v_d, i_{ac}, i_{bd}$. The pair $(s_c, \boldsymbol{\phi})$ is then a controlled-system from s.

Dually, a notion of subsystem, in the behavioral approach, is partially hinted at from the theory of latent variables. It can be generally thought that a subsystem of a big system consists of a projection of the big system onto only the variables of interest. We arrive at the following observation:

Definition 5.3.4 (Subsystem)**.** *Let $\boldsymbol{\phi} : s \Rightarrow s_{sub}$ be a morphism of systems. The pair $(s_{sub}, \boldsymbol{\phi})$ is said to be a subsystem of s, if the components of $\boldsymbol{\phi}$ are surjective maps. We may refer to s_{sub} as the subsystem if $\boldsymbol{\phi}$ is clear from the context.*

Example 5.3.5. *The system underlying the circuit (P_s) can be seen as a subsystem of that of (P):*



Let $p_s : \mathcal{B}_{P_s} \to \mathbb{U}_{P_s}$ and $p : \mathcal{B}_P \to \mathbb{U}_P$ be the systems of (P_s) and (P), respectively. Then $\mathbb{U}_{P_s}$ is the free $\mathbb{R}$-vector space with basis $\{v_{g'}, v_{h'}, i_{g'h'}\}$. The set $\mathcal{B}_{P_s}$ is the subset of $\mathbb{U}_{P_s}$ whose tuples satisfy the laws of the circuit. The morphism $\boldsymbol{\phi} : p \Rightarrow p_s$ is defined uniquely such that ϕ_U sends v_g, v_h, i_{gh} in the basis of $\mathbb{U}_P$ respectively to $v_{g'}, v_{h'}, i_{g'h'}$, *and everything else remaining in the basis of $\mathbb{U}_P$ to* 0*. The pair $(p_s, \boldsymbol{\phi})$ is then a subsystem of p.*

The use of these two notions will appear in the interconnection of systems. Informally, two systems are interconnected by *gluing* them along a common subsystem to yield a controlled-system. This approach will embody the nature of variable sharing stressed at by the behavioral approach.

Remark: Although we may think of the controlled-systems and the subsystems as the domains or co-domains of the morphisms, the notion however is really embedded in the morphism. Indeed, two different morphisms from s to s_{sub} with surjective components yield different subsystems.

Remark: The prefix sub of subsystem typically alludes to a possibility of embedding. It seems counterintuitive that surjective maps rather than injective maps are involved. Similarly, controlled systems would advocate identifying parts of the system together as a means of control. This hints that surjectve maps rather than injective maps are to be in play. Such an unease will be remedied in a future section, by simply *reversing* the direction of the morphisms.

Recovering the fixed point of view.

To recover a fixed point of view, we fix the codomains of our systems. We allow only systems of the form $B \to U$ for a fixed set U, and allow only degenerate morphisms that only map U identically to itself. If s and s' are two systems, we then define a partial-order $s \leq s'$ if, and only if, s factors through s', i.e., $s = s'h$ for some map h. Note that if $s = s'h$ and s is injective, then h is injective. If $s = s'h$ and both s and s' are injective, then h is injective and unique. We then obtain a lattice isomorphic to the lattice of behavior of the universum U. With a fixed universum, the system $B \to U$ is a controlled-system from $B' \to U$ if, and only if, $B \subseteq B'$. The notion of subsystem, however, completely disappears. It appears in hidden form through the theory of latent variables.

5.3.2 Revisiting: latent variables.

In this subsection, we establish that a Willems system $(\mathbb{U}, \mathcal{B})$ is a manifest system of $(\mathbb{U} \times \mathbb{U}_l, \mathcal{B}_f)$ with latent variables if, and only if, $\mathcal{B} \to \mathbb{U}$ is a subsystem of $\mathcal{B}_f \to \mathbb{U} \times \mathbb{U}_l$.

Proposition 5.3.6. *Let $s : B \to U \times U_l$ be a system, and consider a surjective map $U \xrightarrow{\pi} U'$, then (up to isomorphism) there is a unique set B' and a unique surjective map p such that:*

$$\begin{tikzcd}
B \arrow[r, "s"] \arrow[d, "p"] & U \arrow[d, "\pi"] \\
B' \arrow[r] & U'
\end{tikzcd}$$

is a morphism of systems, i.e. is commutative with $B' \to U'$ injective. Furthermore, B' is isomorphic to $im(\pi s)$, the image set of πs.

Proof. Every set map $f : A \to B$ has a unique factorization $f = is$ where s is surjective and i is injective. Indeed, let is and $i's'$ be two factorizations, then $is(A)$ and $i's'(A)$ have the same cardinality. Since i and i' are injective, then $s(A)$ and $s'(A)$ have the same cardinality, and so are isomorphic. As sets, they are isomorphic to $im(f)$. □

Specifying latent variables amounts to specifying a projection map $\mathbb{U}$ onto $\mathbb{U}'$. In particular, the surjective map $\mathbb{U} \times \mathbb{U}_l \to \mathbb{U}$ that forgets the components of $\mathbb{U}_l$ and maps the components in $\mathbb{U}$ identically onto $\mathbb{U}$ automatically identifies the set $\mathbb{U}_l$ as the universum of latent variables. This surjective map induces a unique subsystem by projecting the full behavior onto the manifest behavior.

Corollary 5.3.7. *Let $s : B \to U$ be a system, and $\pi : U \times U_l \to U$ be the projection onto the first coordinate $(u, u_l) \mapsto u$, then $im(\pi s) = \{b \in U : (b, l) \in B$ for some $l\}$.* □

Every Willems system with latent variables uniquely defines its manifest system as a subsystem. However some subsystems cannot be realized as a manifest system of some Willems system of latent variables. The notion of subsystem then properly subsumes the notion of latent variables.

Example 5.3.8. *Referring back to the two-port blackbox abstraction of (P), recall that $\mathbb{U}_{two-port}$ is the free $\mathbb{R}$-vector space with basis $\{v_{e'}, v_{f'}, v_{i'}, v_{j'}, i_{e'}, i_{i'}, i_{f'}, i_{j'}\}$. Define the map $\phi : \mathbb{U}_P \to \mathbb{U}_{two-port}$ to be the projection that sends $v_e, v_f, v_i, v_j, i_{eg}, i_{gi}, i_{fh}, i_{hj}$ in the basis of $\mathbb{U}_P$ respectively to $v_{e'}, v_{f'}, v_{i'}, v_{j'}, i_{e'}, i_{i'}, i_{f'}, i_{j'}$, and everything else remaining in the basis of $\mathbb{U}_P$ to 0. The unique subsystem $\boldsymbol{\phi}$ induced by a ϕ_U component equal to ϕ has the system $\mathcal{B}_{two-port} \to \mathbb{U}_{two-port}$ as a codomain. It thus defines $(\mathcal{B}_{2-port} \to \mathbb{U}_{2-port}, \boldsymbol{\phi})$ as a subsystem of $\mathcal{B}_P \to \mathbb{U}_P$.*

Subsystems will have their use in interconnection of systems. We generally think of the manifest variables as the variables that we wish to model, or rather that are of interest. We may then think of them as being the variables of interest when it comes to interconnecting two systems. More precisely, we can think of them as being the variables that two systems will share. As a pick of variable (i.e., a projection) directly induces a subsystem, we may think of interconnection as sharing a common subsystem. The problem is that two systems may not share a common non-trivial subsystem. There are many pairs of systems s and s', where if $\phi : s \Rightarrow s_{sub}$ and $\phi : s' \Rightarrow s_{sub}$ are morphisms such that (s_{sub}, ϕ) and (s_{sub}, ϕ') are subsystems of s and s', then s_{sub} is the trivial identity map over the set with one element. We can however relax the notion of subsystem to that of a quasi-subsystem. Then every pair of systems would share a non-trivial quasi-subsystem in common.

Definition 5.3.9 (Quasi-Subsystem)**.** *Let $\boldsymbol{\phi} : s \Rightarrow s_{qsub}$ be a morphism of systems. The pair $(s_{qsub}, \boldsymbol{\phi})$ is said to be a quasi-subsystem of s, if the second component ϕ_U of $\boldsymbol{\phi}$ is surjective. We may refer to s_{qsub} as the quasi-subsystem if ϕ is clear from the context.*

5.3.3 Revisiting: interconnection and variable sharing.

The behavioral approach encourages that systems be made from smaller pieces by identifying variables together. We introduce a construct, termed *category*, to aid in capturing the pattern of this idea. Our systems along with their morphisms will form a category. The reason for introducing categories is that interconnection and variables sharing amounts only to an instantiation of a general construction known as *pullback*.

Interlude on categories.

A category can be simply viewed as a directed multi-graph, where the arcs can be composed associatively, and every node has a self-arc that produces no effect when composed.

Definition 5.3.10 (Category). *A category $\mathcal{C}$ consists of:*

i. *A class $Obj(\mathcal{C})$ of objects.*

ii. *A set $Hom_{\mathcal{C}}(A, B)$ of morphisms for every ordered pair (A, B) of objects. A morphism f in $Hom_{\mathcal{C}}(A, B)$ is denoted by $f : A \to B$. If $f : A \to B$ is a morphism, we refer to A and B as the domain and codomain of f.*

iii. *A composition map $Hom(A, B) \times Hom(B, C) \to Hom(A, C)$ for every ordered triple (A, B, C) of objects. The composition of two morphisms $f : A \to B$ and $g : B \to C$ is denoted by either $g \circ f$ or gf.*

vi. *An identity morphism $id_A \in Hom(A, A)$ for every object A.*

This data is subject to two axioms:

A.1. *For every $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have $(f \circ g) \circ h = f \circ (g \circ h)$.*

A.2. *For every $f : A \to B$, we have $f \circ id_A = id_B \circ f = f$.*

The primordial example of a category is the category **Set** where the objects are sets, and morphisms are set functions. Other typical examples may include the category of vector spaces (over a fixed field) with linear maps, the category of topological spaces with continuous maps, the category of groups with group homomorphisms. On a different end, every partially-ordered set forms a category by declaring the elements of the set as the objects, and having $Hom(A, B)$ contain exactly one morphism (and none otherwise) if, and only if, $A \leq B$ in the partial order. Composition, in such a case, reflects the transitive property. The presence of the identity arrow, reflects reflexivity.

Proposition 5.3.11. *The systems (i.e., the injective maps $B \to U$) along with their morphisms (defined in subsection 5.3.1) form a category.*

Proof. The conditions were already verified in section 5.3.1. □

We denote by **System** the category of systems whose objects are injective maps $B \to U$ and morphims ϕ are defined in subsection 5.3.1.

Back to interconnection.

We introduce a universal construction in general categories termed pullback (or fibered-product). Once instantiated to our category **System**, it directly recovers the notion of interconnection and variable sharing.

Definition 5.3.12 (Pullback). *Let $\mathcal{C}$ be a category, and let $f_1 : A_1 \to B$ and $f_2 : A_2 \to B$ be two morphisms in $\mathcal{C}$ with the same codomain. The pullback of (f_1, f_2) consists of a triple (K, p_1, p_2) where K is an object of $\mathcal{C}$ and $p_1 : K \to A_1$ and $p_2 : K \to A_2$ are morphisms such that $f_1 p_1 = f_2 p_2$ satisfying the following universal property: for every other triple (H, q_1, q_2) such that $q_1 f_1 = q_2 f_2$, there is a unique morphism $h : H \to K$ for which the following diagram commutes:*

$$\begin{tikzcd}
H \arrow[drr, bend left, "q_1"] \arrow[dr, dotted, "h"] \arrow[ddr, bend right, "q_2"'] & & \\
& K \arrow[r, "p_1"] \arrow[d, "p_2"] & A_1 \arrow[d, "f_1"] \\
& A_2 \arrow[r, "f_2"] & B
\end{tikzcd} .$$

We refer to K as the object of the pullback*, denoted by $A_1 \times_B A_2$.*

If $f_1 : A_1 \to B$ and $f_2 : A_2 \to B$ are set maps in **Set**, then the pullback (K, p_1, p_2) consists of $K = \{(a_1, a_2) \in A_1 \times A_2 : f_1 a_1 = f_2 a_2\}$. We consider only the object of the pullback in this paper, and generally forget about the remaining maps. The definition of a pullback instantiates to **System** as:

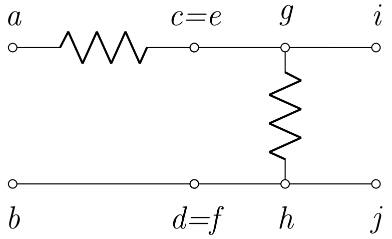
Proposition 5.3.13. *Let $s : B \to U$, $s' : B' \to U'$ and $s_c : B_c \to U_c$ be systems, and let $\boldsymbol{\phi} = (\phi_B, \phi_U) : s \Rightarrow s_c$ and $\boldsymbol{\phi} = (\phi'_B, \phi'_U) : s' \Rightarrow s_c$ be morphisms of systems. Then $s \times_{s_c} s'$ is the system $B^* \to U^*$ where $B^* = \{(b, b') \in B \times B' : \phi_B b = \phi'_B b'\}$, $U^* = \{(u, u') \in U \times U' : \phi_U u = \phi'_U u'\}$ and the set map $B^* \to U^*$ is the restriction of the product map $s \times s' : B \times B' \to U \times U'$ to the domain B^*.*

Proof. Let s^* be $B^* \to U^*$, and $\boldsymbol{\phi}^* : s^* \Rightarrow s \times s'$ be the canonical controlled-system. The pullback is $(s^*, \pi_s \boldsymbol{\phi}^*, \pi_{s'} \boldsymbol{\phi}^*)$ where π_s and $\pi_{s'}$ are the projections from $s \times s'$ to s and s' respectively. Commutativity of diagrams and the universal property can be easily checked, and follow from the case of **Set**. □

We return to variable sharing. Let $s : B \to U \times U_c$ and $s' : B' \to U' \times U_c$ be two systems, and suppose we want to share the variables given by the universum U_c. We then have projections $p : U \times U_c \to U_c$ and $p' : U' \times U_c \to U_c$. We pick $(s_c, \boldsymbol{\phi})$ and $(s_c, \boldsymbol{\phi}')$ to be two arbitrary quasi-subsystems of s and s' respectively, where ϕ_U and ϕ'_U are p and p' respectively. Such quasi-subsystems always exist. For instance, we may trivially pick $s_c = id_{U_c}$, $\boldsymbol{\phi} = (ps, p)$ and $\boldsymbol{\phi}' = (p's', p')$. The interconnected system obtained by sharing the variables in U_c is then given by the pullback of $\boldsymbol{\phi}$ and $\boldsymbol{\phi}'$. Any pair of *common* quasi-subsystems, whose U components are p and p', yields the same interconnected system.

Corollary 5.3.14. *Let $(s_c, \boldsymbol{\phi})$ and $(s_c, \boldsymbol{\phi}')$ be two quasi-subsystems of $s : B \to U \times U_c$ and $s' : B' \to U' \times U_c$ respectively. Suppose further that s_c is a map $B_c \to U_c$ and that ϕ_U and ϕ'_U are the projections $p : U \times U_c \to U_c$ and $p' : U' \times U_c \to U_c$. Then, the object of the pullback of ϕ and ϕ' represents the system $B \times_{B_c} B' \to (U \times U_c) \times_{U_c} (U' \times U_c)$ where the universum corresponds to $U \times U_c \times U'$ and the behavior to $B \times U' \cap U \times B'$.* □

Example 5.3.15. *We define s_c to be $id : \mathbb{U}_c \to \mathbb{U}_c$ where $\mathbb{U}_c$ is the free $\mathbb{R}$-vector space with basis $\{v_{c=e}, v_{d=f}\}$. Let $(s_c, \boldsymbol{\phi})$ and $(s_c, \boldsymbol{\phi}')$ be two quasi-subsystems of $s : \mathcal{B}_S \to \mathbb{U}_S$ and $s' : \mathcal{B}_P \to \mathbb{U}_P$ respectively. The component ϕ_U is defined to send v_c, v_d in the basis of $\mathbb{U}_S$ respectively to $v_{c=e}, v_{d=f}$, and everything else remaining in the basis of $\mathbb{U}_S$ to 0. Similarly, the component ϕ'_U is defined to send v_e, v_f in the basis of $\mathbb{U}_P$ respectively to $v_{c=e}, v_{d=f}$, and everything else remaining in the basis of $\mathbb{U}_P$ to 0. The system corresponding to the circuit:*



is then the object of the pullback of $\boldsymbol{\phi}$ and $\boldsymbol{\phi}'$.

If $\{*\}$ denotes *the* set with one element, then $1 = \{*\} \to \{*\}$ is a system. Furthermore, for every system s, the unique morphism $s \Rightarrow 1$ is a subsystem. In case we pullback from $s \Rightarrow 1$ and $s' \Rightarrow 1$, we are indicating that s and s' are not sharing any variables. The system we recover from their interaction is then simply the two systems independently put together. The system we recover from the pullback is just the product system $s \times s'$ corresponding to the product of the universum and the product of the behavior.

Important remark. Let $\boldsymbol{\phi} : s \Rightarrow s_c$ and $\boldsymbol{\phi} : s' \Rightarrow s_c$ be two morphisms, non-necessarily quasi-subsystems, and denote by (K, π, π') the pullback of $\boldsymbol{\phi}$ and $\boldsymbol{\phi}'$. In that case, π and π' are morphisms of systems, and we may form the morphism $\pi \times \pi' : K \Rightarrow s \times s'$. The pair $(K, \pi \times \pi')$ will always be a controlled-system. Thus variable sharing then consists of pulling back along a common quasi-subsystem to yield a controlled-system on the product of the systems, i.e. on the separate systems simply put next to each other.

Limits. In case we wish to interconnect more than two systems at a time, we can generalize pullbacks to (projective) limit. Such a generalization will not be studied in this paper.

5.3.4 Revisiting: behavioral equations.

Behavioral equations consists of two morphisms with the same domain and codomain. The behavior they represent then corresponds to their equalizer.

Definition 5.3.16 (Equalizer)**.** *Let $\mathcal{C}$ be a category, and $f, g : A \to B$ two morphisms. An equalizer of f and g is a pair $(E, e : E \to A)$ such that $fe = ge$ and for every map $h : H \to A$ such that $fh = gh$ there is a unique map $u : H \to E$ such that $h = eu$.*

Every pair of maps between two sets have an equalizer. It is the subset of A that is sent to the same image in B by both maps. As such, the equalizer will always be injective. We can immediately recognize that the behavior of the system is the equalizer to the behavioral equations. Therefore:

Proposition 5.3.17. *The triple $(\mathbb{U}, \mathbb{E}, f_1, f_2)$ is a* behavioral equation representation *of $(\mathbb{U}, \mathcal{B})$ if and only if $(\mathcal{B}, \mathcal{B} \to \mathbb{U})$ is the equalizer of f_1 and f_2.* □

However, more structure can be harvested. Let us define a category **Equation** of behavioral equation representations. The objects of **Equation** are pairs (f_1, f_2) of set maps $U \to E$ for every pair of sets U and E. A morphism $\boldsymbol{\psi}$ from $(f_1, f_2) : U \to E$ to $(g_1, g_2) : U' \to E'$ is also a pair of maps (ψ_U, ψ_E) with the commutativity properties $\psi_E f_1 = g_1 \psi_U$ and $\psi_E f_2 = g_2 \psi_U$.

One should think of **Equation** as a category of syntax, i.e. of descriptions of systems. This parallels **System** which acts as the category of semantics. Pullbacks in the category **Equation** instantiates to constructing our systems syntactically. Indeed, pullbacks in **Equation** consist of stacking the equations together while making sure that common variables are well identified and taken care of. We will not flush out the details of this. The two categories are related through the equalizer rule. We make this precise by introducing the notion of a functor.

Interlude on functors.

A functor is a mapping that preserves the structure of a category. If categories are viewed as directed multi-graphs, then functors are graph homomorphisms that preserve composition and identity.

Definition 5.3.18. *A functor $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a rule that assigns an object $F(C)$ of $\mathcal{D}$ to every object C of $\mathcal{C}$, and a morphism $F(f) : F(C_1) \to F(C_2)$ in $\mathcal{D}$ to every morphsim $f : C_1 \to C_2$ in $\mathcal{C}$. We require a functor to preserve the identity morphisms, i.e., $F(id_C) = id_{F(C)}$ and composition, i.e., $F(f \circ g) = F(f) \circ F(g)$.*

Consider, as an example, the power-set functor that sends a set A to 2^A its set of subsets. It also sends functions $A \to B$ to functions $2^A \to 2^B$ by mapping a subset of A to its image set. It forms a functor from **Set** to **Set**. On a different end, every order-preserving map between partially-ordered sets is a functor on the induced categories.

Back to the behavioral equations.

Let **Arr-Eq** be the map that sends an object in Obj(**Equation**) to the morphism component in its equalizer in Obj(**System**).

Proposition 5.3.19. *The map* **Arr-Eq** *lifts to a functor from **Equation** to **System**.*

Proof. If $(\psi_U, \psi_E) : (f_1, f_2) \Rightarrow (g_1, g_2)$ is a morphism in **Equation**, we let $\textbf{Arr-Eq}(\psi_U, \psi_E)$ be the unique morphism $\boldsymbol{\phi} : \textbf{Arr-Eq}(f_1, f_2) \Rightarrow \textbf{Arr-Eq}(g_1, g_2)$ in **System** such that $\phi_U = \psi_U$. □

The crucial property that connects syntax and semantics well is that **Arr-Eq** preserves pullbacks. The semantics reflect interconnection in the syntax.

Proposition 5.3.20. *Let $\boldsymbol{\psi} : e \Rightarrow e_c$ and $\boldsymbol{\psi}' : e' \Rightarrow e_c$ be morphisms in* **Equation***, then* **Arr-Eq** *takes the object of the pullback of $(\boldsymbol{\psi}, \boldsymbol{\psi}')$ to the object of the pullback of $(\textbf{Arr-Eq}\boldsymbol{\psi}, \textbf{Arr-Eq}\boldsymbol{\psi}')$.*

Proof. Let $(e^*, \boldsymbol{\psi}^*, \boldsymbol{\psi}'^*)$ be the pullback of $(\boldsymbol{\psi}, \boldsymbol{\psi}')$. As equalizers preserve limits, and in particular pullbacks, the domain B^* of $\textbf{Arr-Eq}e^*$ coincides with the domain of the object of the pullback of $(\textbf{Arr-Eq}\boldsymbol{\psi}, \textbf{Arr-Eq}\boldsymbol{\psi}')$. The codomains trivially coincide, and are denoted by U^*. Finally there is a unique set map $B^* \to U^*$ ensuring that the emerging diagram commutes. □

We first illustrate with a clear example.

Example 5.3.21. *We consider two resistors (R) and (R'), and define a syntactical interconnection, on the level of behavioral equation representations.*

The universa $\mathbb{U}_R$ and $\mathbb{U}_{R'}$ are the $\mathbb{R}$-vector spaces with basis $\{v_a, v_b, i_{ab}\}$ and $\{v_c, v_d, i_{cd}\}$, respectively. A behavioral equation representation of (R) consists of a pair $(f_R, 0)$ where $f_R : \mathbb{U}_R \to \mathbb{R}$ sends v_a, v_b, i_{ab} to $-1, 1, R$ respectively. Likewise, a behavioral equation representation of (R') consists of a pair $(f_{R'}, 0)$ where $f_{R'} : \mathbb{U}_{R'} \to \mathbb{R}$ sends v_c, v_d, i_{cd} to $-1, 1, R'$ respectively. Let $\mathbb{R}^2$ be the free $\mathbb{R}$-vector space with basis $\{v, i\}$ and let $0_2 : \mathbb{R}^2 \to 0$ be the zero map. We can uniquely define morphisms $\boldsymbol{\psi} : (f_R, 0) \Rightarrow (0_2, 0_2)$ and $\boldsymbol{\psi}' : (f_{R'}, 0) \Rightarrow (0_2, 0_2)$ such that $\psi_U : \mathbb{U}_R \to \mathbb{R}^2$ sends v_a, v_b, i_{ab} to $0, v, i$ and $\psi'_U : \mathbb{U}_{R'} \to \mathbb{R}^2$ sends v_c, v_d, i_{ab} to $v, 0, i$. Pulling back along $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$ yields as object the pair $(f, 0) : \mathbb{R}^4 \to \mathbb{R}^2$ which is a behavioral equation representation for the series circuit once (R) and (R') are interconnected at b and c.

We may now link to the running example:

Example 5.3.22. *Let us suppose that $\mathbb{R}^2$ has basis $\{e_1, e_2\}$. A behavioral equation representation of (S) can be identified with a linear map $f_S : \mathbb{U}_S \to \mathbb{R}^2$, that sends $v_a, v_b, v_c, v_d, i_{ac}, i_{bd}$ to $-e_1, -e_2, e_1, e_2, -Re_1, 0$. Clearly, the kernel of f_S gives $\mathcal{B}_S$. Let f_c be the identity map $id_2 : \mathbb{R}^2 \to \mathbb{R}^2$. We may now construct the unique morphism $\boldsymbol{\psi} : (f_S, 0) \to (id_2, 0)$ such that ψ_E is the identity map. A behavioral representation of (P) can be identified with a linear map $f_P : \mathbb{U}_P \to \mathbb{R}^9$, and a canonical morphism $\boldsymbol{\psi}' : (f_P, 0) \to (id_2, 0)$ may be set up. The object of the pullback along $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$ yields a behavioral equation representation of the interconnection of (S) and (T) by identifying c with e and d with f.*

Important Remark: It is very desirable that each system possesses a simplest behavioral equation representation of it. The presented syntax category along with the **Arr-Eq** functor will not allow such a thing. Every system will have many behavioral equations attached to it, but no *universal* one. We can however get a much better category and functor by working with generalized systems, where we drop the injectivity condition on the maps $B \to U$. Each generalized system then has a *simplest* description to it. The new category preserves all the features described in this section, but consists of a more involved construction. To preserve flow of the paper, we present the desired category in an appendix section on generalized systems.

5.4 The big picture.

We constructed two categories. A category of syntax **Equation**, reflecting syntactical descriptions of systems through behavioral equations, and a category of semantics **System**, reflecting the objects behind the equations, the behaviors. Those two categories are linked by an interpretation functor **Arr-Eq** that sends each description to the underlying system it described. Interconnection of systems in **System** consist of taking pullbacks. We uncovered a notion of (quasi-)subsystem, and a notion of controlled-system. Indeed, variable sharing consists of pulling back along a shared quasi-subsystem to get a controlled-system. Syntactical constructions in **Equation** also consist of taking pullbacks. Finally, the interpretation functor **Arr-Eq** preserves pullbacks, thus translating syntactical interconnection to semantical interconnection.

Throughout the development, sub-systems were identified with surjective maps, and controlled-systems with injective maps. Such an association might seem unnatural as we already described in subsection 5.3.1. Indeed, the prefix sub of subsystem typically alludes to a part of bigger system, namely a possibility of embedding. It then seems counter-intuitive that surjective maps rather than injective maps are involved. Also, controlled systems lead us to think of identifying parts of the system as a means of control. This identification is intuively percieved through surjective maps. To recover a more intuitive association, we then only need to flip the direction of the morphisms. Such a flip is discussed in the next sub-section.

5.4.1 The opposite approach.

Rather than thinking of a Willems system as a pair $(\mathbb{U}, \mathcal{B})$ with $\mathcal{B} \subseteq \mathbb{U}$, we will think of it as a pair $(2^{\mathbb{U}}, 2^{\mathcal{B}})$, where 2^S denotes the set of subsets of a set S. Then, from the relative point of view, a system is no longer an injective map $B \to U$, but a surjective map $2^U \to 2^B$ that sends $S \in 2^U$ to $S \cap B$. Such a consideration might seem absurd, but it is equally valid and may appear to be more natural. Subsystems are now associated with injective morphisms, and controlled-systems are now associated with surjective morphisms. Overall, this transformation only reverses the *direction* of the morphisms involved.

Let us define **Bool** to be the category whose objects are complete atomic boolean lattices, and whose morphisms are the lattice homomorphisms. Every complete atomic boolean lattice is isomorphic to 2^S for some set S. We may then just consider the objects as lattices of the form 2^S for some set S. The categories **Set** and **Bool** may be related through two functors.

i. **From Set to Bool.** If $f : S \to T$ is a set map, then $f^{-1} : 2^T \to 2^S$ is a lattice homomorphism. We denote by $F : \mathbf{Set} \to \mathbf{Bool}$ the functor that sends S to 2^S and $f : S \to T$ to $f^{-1} : 2^T \to 2^S$.

ii. **From Bool to Set.** If $\phi : 2^T \to 2^S$ is a lattice homomorphism, then there exists a unique map $G\phi : S \to T$ such that $(G\phi)^{-1} = \phi$. Indeed, as $\phi\{t\} \cap \phi\{t'\} = \emptyset$ for every $t \neq t'$ in T, every s belongs to a set $\phi\{t\}$ for a unique t. If the map $G\phi$ sends every $s \in \phi\{t\}$ to t, then $(G\phi)^{-1} = \phi$. We denote by $G : \mathbf{Bool} \to \mathbf{Set}$ the functor that sends 2^T to T and ϕ to $G\phi$ as just described.

We make the relation between **Bool** and **Set** precise by defining the notion of an opposite category.

Definition 5.4.1 (Opposite Categories)**.** *The opposite category $\mathcal{C}^{op}$ of $\mathcal{C}$ is the category containing the objects of C, and a morphism $f^{op} : B \to A$ for every $f : A \to B$ in C. If $f^{op} : B \to A$ and $g^{op} : C \to B$ are in $\mathcal{C}^{op}$, then $f^{op} \circ g^{op} = (g \circ f)^{op}$.*

The notion of opposite categories, in general, allows us to dualize both definitions and results.

Definition 5.4.2 (Pushouts)**.** *The pushout of $f : A \to B$ and $g : A \to C$ in $\mathcal{C}$ is the pullback of f^{op} and g^{op} in $\mathcal{C}^{op}$.*

Definition 5.4.3 (Coequalizers)**.** *The coequalizer of $f_1 : A \to B$ and $f_2 : A \to B$ in $\mathcal{C}$ is the equalizer of f^{op} and g^{op} in $\mathcal{C}^{op}$.*

The key fact is that **Bool** is *equivalent* $\mathbf{Set}^{op}$. Such a fact lifts to our category of systems. We shall not formalize the term *equivalent*, but rather describe what it entails for us. For all purposes, working in **Bool** is the same as working in $\mathbf{Set}^{op}$. The functors F and G give us the mean to flip the arrows. The equivalence manifests itself first in:

Proposition 5.4.4. *A map f is injective (resp. surjective) in* **Set** *if, and only if, $Ff = f^{-1}$ is surjective (resp. injective) in* **Bool**.

Proof. The result follows from the definition of F. □

We can then define a category **Bool-System** to have, as objects, surjective boolean lattice homomorphisms $2^U \to 2^B$. A morphism $\boldsymbol{\psi}$ in **Bool-System** is then a pair of homomorphisms (ψ_U, ψ_B) such that the diagram:

$$\begin{tikzcd}
2^U \arrow[r, "h"] \arrow[d, "\psi_U"] & 2^B \arrow[d, "\psi_B"] \\
2^{U'} \arrow[r, "h'"] & 2^{B'}
\end{tikzcd}$$

commutes, i.e., such that $h'\psi_U = \psi_B h$.

Building on proposition 5.4.4, one can check that h (resp. Fs) is an object of **Bool-System** if, and only if, Gh (resp. s) is an object of **System**. A morphsim

$\boldsymbol{\psi} = (\psi_U, \psi_B)$ is a morphim of **Bool-System** if, and only if, $G\boldsymbol{\psi} = (G\psi_B, G\psi_U)$ is a morphism of **System**. Most importantly:

Proposition 5.4.5. *The bool-system (h, i, i') is the pushout of $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$ if, and only if, (Gh, Gi, Gi') is the pullback of $G\boldsymbol{\psi}$ and $G\boldsymbol{\psi}'$.*

Proof. As **Bool** is equivalent to $\mathbf{Set}^{op}$, the category **Bool-System** is equivalent to the category $\mathbf{System}^{op}$. The result follows by definition. □

We are only translating diagrams in **System** to diagrams in **Bool-System** without losing any property at all.

A subsystem in **Bool-System** is then a morphism where both components are injective. A controlled-system in **Bool-System** is then a morphism where both components are surjective. A part of a universum is also now described as an embedding (i.e., injection) rather than a projection (i.e., surjection). The intuition is thus remedied. One way to construct topological spaces consists of gluing simple ones together by identifying subspaces. Such a construction is an instantiation of the pushout. Similarly, we may construct complicated systems from simple ones, by identifying subsystems together along a part of the universum, to get a controlled-system.

Discussion and Remarks.

A system restricts (or excludes) some outcomes from the universum to yield the behavior. The map $2^{\mathbb{U}} \to 2^{\mathcal{B}}$ can then be seen as the object that encodes the "restrictive capability" of the system. It acts similarly to the equations defining a system, but it is actually independent of the particular (behavioral) equation representation. A rough analogy to this duality would be the duality between varieties and polynomial ideals. The variety is a geometric object encoding the solution, and the polynomial ideal (or the affine coordinate ring) is an algebraic object encoding the equations.

Furthermore, an epi from $2^{\mathbb{U}}$ to $2^{\mathcal{B}}$ induces a kernel (dual to a closure) operator on the subsets of $\mathbb{U}$. For every subset V of outcomes in $\mathbb{U}$, a Willems' system defines which outcomes of V are possible. The behavior of the system can also be recovered from $B = Hom(2^B, 2^*)$. A subsystem is then an objects that is a sub-restriction, whereas a controlled system is an object that is a over-restriction. Again, the object $2^{\mathbb{U}} \to 2^{\mathcal{B}}$ is independent of the particular *syntax*.

We believe this line of thought for the behavioral approach has been well treated using D-module theory through algebraic analysis. The link will not be investigated in this paper, we instead refer the reader to [Qua10] and [Obe90] for an initial thread.

5.4.2 How essential is the category System?

The systems in **System** are defined as injective maps $B \to U$. We may just keep the domain of the map, thus keeping only the behavior of the systems. We definitely lose information. Indeed if $B \subset U$ and $B' \subset U$ are distinct subsets of U, we will have no way of distinguishing them in case they have the same cardinality if we forget the map. Nevertheless, forgetting that information preserves constructions of systems, i.e. pullbacks, and may be a valid definition of systems for some instances.

This idea however suggests adding more information, rather than forgetting some, for additional effects. Given that universa are relative, we may decide to make some elements of the universa distinguished. We may further consider only sets with additional structure, e.g. vector spaces or in general modules over rings. Those objects inherit the same notion of interconnection, and a development may be carried out along the same line as that carried out here. Thus, any *content* of the category is valid, as long as we can have a good interpretation for it as a system. It will nevertheless make a technical difference. However, the intrinsic pattern of interconnection and variable sharing, as well as the pattern of the other ideas we have described, will remain unchanged.

Remark: Categories other than **Equation** may also act as a category of syntax. Following the lines of the running example, we should expect circuit diagrams, if formalized properly, to act as an alternative category of syntax, when our systems are restricted to underly resistive circuits.

5.5 The concluding picture.

We abstract away to the following scheme:

$$\begin{tikzcd}\textbf{System Syntax} \arrow[r, "Interp"] & \textbf{System Semantics}\end{tikzcd}$$

Constructions both in the syntax and semantics category consists of taking pushouts, and the functor **Interp** preserves (or commutes with) pushouts. Pushouts may replace pullbacks, without loss of generality, by working with the opposite categories. Pushouts may be generally replaced by colimits (or inductive limits), and then **Interp** would then be required to preserve colimits. In general categories, the role played by injective (resp. surjective) maps in **Set** will be played by monic (resp. epic) morphism. Thus subsystems would correspond to monos, and controlled-systems to epis. A mono $A \to B$ defines A as a subobject of B, while an epi $B \to C$ defines C as a quotient-object of B. In general, additional properties would be required by the monos and epis, e.g., regularity, to suit our needs. We refer to [ML98] for further details on functorial matters.

The functor **Interp** is envisioned to have more properties than what will be explicitly mentioned in this paper. For instance, in our setting, every system admits at least one behavioral equation representation, or description. It is very pleasing to have every system have a *simplest* description. The syntax category thus provided does not afford that. We may construct one that does by going to generalized systems. When such a situation happens, **Interp** is said to have an adjoint.

5.5.1 In the case of partially ordered sets.

Every partially ordered set (termed *poset*) forms a category by declaring the elements of the set as the objects, and having $Hom(A, B)$ contain one morphism if, and only if, $A \leq B$ in the partial order. A functor between two categories induced by posets is then only an order-preserving map.

We consider, in this subsection, only posets where every pair of elements admits a least upper bound. The operation of taking least upper bounds can be declared as a binary operation $\vee$, termed *join*. The algebras obtained are then termed join-semilattices. Taking a pushout along two morphims (when posets are viewed as categories) consists of taking the join (in the join-semilattice) of the their codomains.

We may then consider the categories **System Syntax** and **System Semantics** to be induced by posets. The functor **Interp** is then an order-preserving map that commutes with the join operation. If $s \leq s'$, then s is a subsystem of s' and s' is a controlled-system from s. We can acquire a physical interpretation if we consider **System Semantics** to be the lattice of behaviors over a fixed universum.

5.5.2 Intended application.

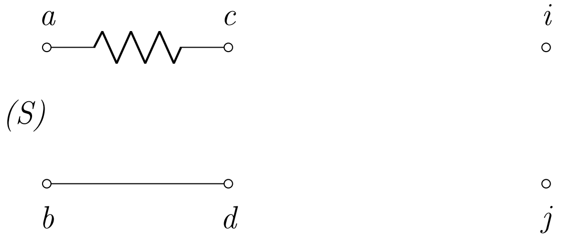
We return to our main concern of uncovering phenomena that emerge from the interaction of systems. A theory of interconnection cannot be enough to account for interaction-related effects. Interconnecting two systems can only yield an interconnected systems. Such effects may only emerge once we decide to focus on a feature or a property of the system, that we term *phenome*.

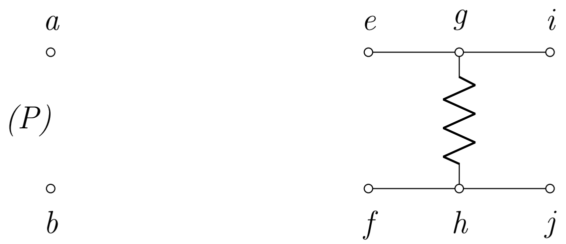
We generally arrive at a phenome by forgetting, or concealing, information from the system. We may then think of the phenome as a simplified system. Phenomes then live in a category and inherit a notion of interconnection through pushouts. The situation is summarized as:

$$\begin{tikzcd}\textbf{Phenome} & \textbf{System} \arrow[l, "Forget"']\end{tikzcd}$$

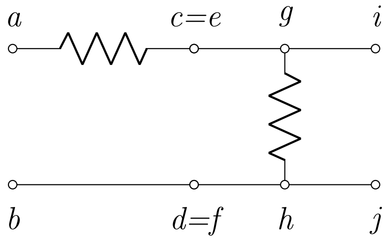
Whether or not new phenomena emerge upon the interaction of systems is now encoded in the functor **Forget**. New phenomes can emerge precisely when the **Forget** functor does not commmute with pushouts. Indeed, the irrelevant things that we had forgotten actually come together and produce new observables. If **Forget** always commutes with pushouts, then the phenome of the interconnected system is simply the interconnection of the phenomes of the separate systems.

Example 5.5.1. *We reconsider the circuits (S) and (P) and augment them with* external *terminals:*





Each of the behavior underlying (S) and (P), when restricted to the variables v_a, v_b, v_i and v_j, consists of a four dimensional $\mathbb{R}$-vector space. When we interconnect (S) and (P) by identifying c to e and d to f, we obtain:



The behavior of the underlying interconnected system, restricted again to the variables v_a, v_b, v_i and v_j, now consists of a one dimensional $\mathbb{R}$-vector space. The internal *mechanisms (declared* internal *by focusing only on the terminals a, b, i and j) in the circuits interact so as to produce new observables.*

Using methods from homological algebra, we can then relate the phenome of the interconnected system to that of its subsystems, despite the presence of cascade-like effets. We refer the reader to [Ada17a] for the details.

5.6 Appendix: Generalized systems.

Our systems were thus far injective set maps $B \to U$. We may drop the injectivity requirement and obtain the notion of a generalized system, which is only a set map. We then establish a category **Generalized-System** of generalized systems whose objects are set maps $C \to U$, and morphisms $\boldsymbol{\phi}$ between $g : C \to U$ and $g' : C' \to U'$ are pairs of maps (ϕ_C, ϕ_U) such that the diagram:

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "\phi_C"] & U \arrow[d, "\phi_U"] \\
C' \arrow[r, "g'"] & U'
\end{tikzcd}$$

commutes, i.e., such that $\phi_U g = g' \phi_C$.

Given a generalized system $g : C \to U$, we recover our regular interpretation of a system by reading its image, i.e., the injective map $g(C) \to U$. The rule mapping g to $g(C) \to U$ defines a functor from **Generalized-System** to **System**. Most of the notions defined earlier extend naturally to **Generalized-System** while remaining intact on generalized-systems that are injective, i.e., the *regular* systems.

We now define our syntax category **Generalized-Equation**. The objects are pairs $\boldsymbol{\phi}^1, \boldsymbol{\phi}^2 : g \to g'$ of morphisms in **Generalized-System** with the same domain and codomain. A morphism from $(\boldsymbol{\phi}^1, \boldsymbol{\phi}^2)$ to $(\boldsymbol{\psi}^1, \boldsymbol{\psi}^2)$ consist of four set maps $\tau_1, \tau_2, \tau_3, \tau_4$ such that the following diagram commutes for both values of i:

$$\begin{tikzcd}
C \arrow[rrr, "g"] \arrow[dr, "\tau_1"] \arrow[ddd, "\phi^i_C"] & & & U \arrow[dl, "\tau_2"] \arrow[ddd, "\phi^i_U"] \\
& D \arrow[r, "h"] \arrow[d, "\psi^i_D"] & V \arrow[d, "\psi^i_V"] & \\
& D' \arrow[r, "h'"] & V' & \\
C' \arrow[rrr, "g'"] \arrow[ur, "\tau_3"] & & & U' \arrow[ul, "\tau_4"]
\end{tikzcd} .$$

We may now define a functor **Obj-Eq** : **Generalized-Equation** $\to$ **Generalized-System** that sends a pair $\boldsymbol{\phi}^1, \boldsymbol{\phi}^2 : g \to g'$ to the object of its equalizer.

The objects of the category **Equation** can be found in **Generalized-Equation** as follows:

Proposition 5.6.1. *Let* $(\boldsymbol{\phi}^1, \boldsymbol{\phi}^2)$ *be an object in* **Generalized-Equation***, where* $\boldsymbol{\phi}^1$ *and* $\boldsymbol{\phi}^2$ *correspond to the following diagrams, respectively:*

$$\begin{tikzcd}
U \arrow[r, "id_U"] \arrow[d, "f_1"] & U \arrow[d, "*"] \\
E \arrow[r, "*"] & \{*\}
\end{tikzcd} \qquad \begin{tikzcd}
U \arrow[r, "id_U"] \arrow[d, "f_2"] & U \arrow[d, "*"] \\
E \arrow[r, "*"] & \{*\}
\end{tikzcd} .$$

Then **Obj-Eq**$(\boldsymbol{\phi}^1, \boldsymbol{\phi}^2)$ *is isomorphic to* **Arr-Eq**(f_1, f_2). □

The functor **Obj-Eq** is right adjoint to the diagonal functor that sends the generalized-system g to the generalized-equation (id_g, id_g). Every system $C \to U$ then has a *universal* description of the form:

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "id_C"] & U \arrow[d, "id_U"] \\
C \arrow[r, "g"] & U
\end{tikzcd} \qquad \begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "id_C"] & U \arrow[d, "id_U"] \\
C \arrow[r, "g"] & U
\end{tikzcd} .$$

The functor **Obj-Eq** then preserves pullbacks, and more generally (projective) limits.

Chapter 6

Interconnection and memory in linear time-invariant systems

Abstract

We characterize the role played by memory when linear time-invariant systems interact. This study is of interest as the phenomenon that occurs in this setting is arguably the same phenomenon that governs cascading failure and contagion effects in interconnected systems. We aim to later extend solutions presented in this chapter to problems in other desired settings.

The characterization relies on basic methods in homological algebra, and is reminiscent of the rank-nullity theorem of linear algebra. Interconnection of systems is first expressed as an exact sequence, then loss of memory causes a loss of exactness, and finally exactness is recovered through specific algebraic invariants of the systems that encode the role of memory. We thus introduce a new invariant, termed *lag*, of linear time-invariant systems and characterize the role of memory in terms of the lag. We discuss properties of the lag, and prove several results regarding the characterization.

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6.1 Introduction

Consider the following two discrete-time dynamical systems A and B given by the difference equations:

$$(\mathrm{A}) \quad x[n] = y[n] + y[n-1] \quad \text{and} \quad (\mathrm{B}) \quad y[n] = x[n].$$

The signals are real-valued and causal, i.e., $x[n] = y[n] = 0$ for $n < 0$. Although the signals are labeled with matching letters, the systems A and B are distinct and independent. We will be interconnecting them later on, and using matching letters proves to be convenient. One may, although unnecessary, think of A (resp. B) as having input y (resp. x) and output x (resp. y). Our goal is to understand the role that memory plays when the systems A and B interact. This goal is of interest as the phenomenon that occurs in this setting is arguably the same phenomenon that governs cascading failure and contagion effects in interconnected systems. Solutions to problems posed in this setting, with more work, will evolve to solutions to problems in other desired settings.

We may devise a simple experiment to capture this role. We will quickly describe it in a very informal manner, and then expound and formalize it afterwards throughout this introduction. The experiment consists of two steps. In the first step, we allow A and B to naturally interact, forming a combined system A&B, and then forget that A&B has any capacity for memory, to get $(\mathrm{A}\&\mathrm{B})_{mem}$. In the second step, we first forget each of A's and B's capacity for memory, to get A_{mem} and B_{mem}, and then let the memoryless systems interact to form the combined system $\mathrm{A}_{mem}\&\mathrm{B}_{mem}$. We now compare the combined system obtained through the two steps. If $(\mathrm{A}\&\mathrm{B})_{mem}$ and

A$_{mem}$&B$_{mem}$ are the same system, then memory plays no role during interaction. If they are not the same system, then memory certainly has its role.

Formally, suppose we keep from each of A and B only the set of pairs $(x[0], y[0])$ that can appear through trajectories satisfying the respective dynamics, i.e. the set $\{(x[0], y[0]) : x[n] = y[n]+y[n-1]\}$ from A and the set $\{(x[0], y[0]) : y[n] = x[n]\}$ from B. In doing so, we inherently forget (or destroy) all the potential a system possesses to remember the past. Such a set would correspond to the memoryless system that can best explain the trajectories of its respective system. The systems A$_{mem}$ and B$_{mem}$ may then be respectively identified with those two sets. In particular, the pairs that appear in A$_{mem}$ correspond to pairs of the form (r, r) for $r \in \mathbb{R}$. Those pairs generate the trajectories coming from A if we assumed that A is memoryless, i.e. that A is instead given by the equation $x[n] = y[n]$ where $y[n-1]$ is dropped. Similarly, as B is already memoryless, we also recover all pairs of the form (r, r) for $r \in \mathbb{R}$. We now interconnect A and B allowing them to interact by identifying signals together. In our case, the signal x in A is identified with x in B, and y in A is identified with y in B. We get the system A&B:

$$\begin{aligned} x[n] &= y[n] + y[n-1] \\ y[n] &= x[n] \end{aligned}$$

We see that no pair $(x[0], y[0])$ other than $(0, 0)$ can be observed in A&B. Namely, the system (A&B)$_{mem}$ corresponds to $\{(0, 0)\}$. However, the separate pieces A$_{mem}$ and B$_{mem}$ would have naively informed us to expect any pair (r, r) to be observed. Indeed, the system A$_{mem}$&B$_{mem}$ corresponds to $\{(r, r) : r \in \mathbb{R}\}$. This discrepancy lets us conclude that memory plays at least some role. We may then ask ourselves: how big of a role does it play? And, would an answer to such a question be even useful in understanding the operation of the interconnected system?

Let $\mathcal{B}_A$ and $\mathcal{B}_B$ denote the behaviors of the systems A and B, namely the set of trajectories (x, y) that are allowed through A and B respectively. If $\mathcal{B}_{A\&B}$ denotes the behavior of A&B, then we have $\mathcal{B}_{A\&B} = \mathcal{B}_A \cap \mathcal{B}_B$, where $\cap$ is set intersection. We are only casting the example into Jan Willems' behavioral approach to systems theory (see e.g., [PW98] and [Wil07]). If we define $\Phi(\mathcal{B}) := \{(x[0], y[0]) : (x, y) \in \mathcal{B}\}$ to be the projection of $\mathcal{B}$ on time 0, we then get:

$$\Phi(\mathcal{B}_{A\&B}) \neq \Phi(\mathcal{B}_A) \cap \Phi(\mathcal{B}_B).$$

The inequality points out to the discrepancy mentioned above. It thus lets us conclude that memory plays at least some role. The extent of the role played by memory is captured by how unequal the two sides are. Of course, in general, both A and B may be arbitrary linear time-invariant systems.

The work in this chapter enables us to characterize how unequal those two sides are, and link the quantity $\Phi\mathcal{B}_{A\&B}$ to both $\Phi\mathcal{B}_A$ and $\Phi\mathcal{B}_B$ through such a characterization. The characterization relies on basic methods in homological algebra, and is reminiscent of the rank-nullity theorem of linear algebra. Interconnection of systems is first expressed as an exact sequence, then destruction of memory causes a loss of exactness, and finally the inexact sequence is made exact via specific algebraic invariants of the systems at hand.

As a follow up, the quantity $\Phi\mathcal{B}_{A\&B}$ is necessarily a linear subspace of $\mathbb{R}^2$. We know that it is the 0 subspace. It loses one dimension out of two from the fact that $\mathcal{B}_A$ and $\mathcal{B}_B$ are the same and isomorphic to $\mathbb{R}$. It loses the second dimension due to memory. Let $z^{-1}x$ denote the shifted signal where $z^{-1}x[n] = x[n-1]$. If we consider $\mathcal{B}_A + \mathcal{B}_B = \{t_A + t_B : t_A \in \mathcal{B}_A, t_B \in \mathcal{B}_B\}$, then $\Phi\mathcal{B}_{A\&B}$ will lose its second dimension because the linear space $\{t : z^{-1}t \in \mathcal{B}_A + \mathcal{B}_B\}$ has dimension 1 when quotiented by $\mathcal{B}_A + \mathcal{B}_B$. We leave the justification for the content.

6.1.1 Why memory?

This effort falls within a grander effort to understand interconnection of systems and the interaction-related phenomena such interconnections produce. The situation that occurs when we forget memory is the same, on the appropriate abstract level, as the phenomenon observed in contagion or cascading failure related problems. See for instance Appendix 6.8 for one concrete example, and [Ada17c] for more details on that example. Indeed the distinctive property of the phenomenon in such problems can be cast into an inequality similar to that evoked above. All those problems share the same abstract mathematical structure, and we aim to extend the methods and solutions provided here to those settings. We refer the reader to [Ada17a] for more details.

6.1.2 Why linear time-invariant systems?

We consider in this chapter only linear time-invariant systems for at least three reasons. First, if some general theory were to be established, it would require a notion of system and a notion of interconnection. In our linear world, those notions can correspond perfectly to those advocated by the behavioral approach to systems theory. This alleviates us from introducing new unfamiliar ideas on that front. Second, linear time-invariant system are familiar objects, and it is our hope that the phenomena presented (in terms of inequality) would not seem too elusive and can be quickly made to be familiar. Our goal is to focus on the nature of the solution we get in such problems. Third, homological algebra is done most directly through *abelian* objects, e.g. vector spaces or abelian groups. Linear systems theory provide us with such objects without much effort. The behaviors will be modules over rings of formal power series.

6.1.3 Our contribution.

Our contribution is set to answer the following two questions. First, if we are to forget a system's capacity for memory, what piece of information (and in what form) should we retain to characterize the role that would have been played by memory during interaction? Second, how can we use that information to uncover memory related phenomena that occur? The contribution may be summarized in three steps:

i. We link interconnection of linear time-invariant systems to short exact sequences of modules.

ii. We pin the elusive role played by loss of memory as a certain loss of exactness.

iii. We mend the loss of exactness by extracting algebraic objects from the systems, and then constructing a long exact sequence.

The extracted algebraic objects are the pieces of information that encode the role of memory during interaction. The long exact sequence then uses those algebraic objects to relate $\Phi\mathcal{B}_{A\&B}$ to both $\Phi\mathcal{B}_A$ and $\Phi\mathcal{B}_B$. As the work aims to make sense of the inequality:

$$\Phi(\mathcal{B}_{A\&B}) \neq \Phi(\mathcal{B}_A) \cap \Phi(\mathcal{B}_B),$$

we introduce a new invariant, termed lag, of linear time-invariant systems and recover an equality of the form:

$$(\Phi(\mathcal{B}_A) \cap \Phi(\mathcal{B}_B))/\Phi(\mathcal{B}_{A\&B}) = \text{lag}(\mathcal{B}_A + \mathcal{B}_B)/(\text{lag}(\mathcal{B}_A) + \text{lag}(\mathcal{B}_B)).$$

We discuss properties of lag and prove several results related to the characterization.

One may, naturally, forget only delays of length at least T, by keeping only information on signals up to time $T-1$. Specifically, we keep, from A and B, the pairs $\{(x[0,\cdots,T-1], y[0,\cdots,T-1])\}$ instead of $\{(x[0], y[0])\}$. The same phenomenon occurs, and the same techniques and solution apply.

6.2 Mathematical Preliminaries

We assume the reader is familiar with the notions of an abelian group and a commutative ring with unit. Let R be a commutative ring with unit, denoted by 1. An R-module is an abelian group $(M, +)$ with an operation $\cdot : R \times M \to M$ such that $+$ and $\cdot$ distribute over each other, $(rs) \cdot m = r \cdot (s \cdot m)$ for $r, s \in R$ and $m \in M$, and $1 \cdot m = m$ for all $m \in M$. If k is a field, then a k-module is a vector space over k. Let M and N be R-modules, their direct sum $M \oplus N$ is the module consisting of pairs (m, n) for $m \in M$ and $n \in N$. The direct sum induces a canonical injection $i : M \to M \oplus N$ mapping m to $(m, 0)$, and a canonical projection $p : M \oplus N \to M$ mapping (m, n) to m. A module is said to be free if it is a direct sum of copies of R. If M is a module and $r \in \mathbb{R}$ an element, then $rM = \{rm : m \in M\}$. Let N be an R-submodule of M, then the quotient module M/N is the module whose elements are equivalence classes of the form $m + N$ with $m \in M$. We refer the reader to [AM69] for more details on modules and other algebraic concepts.

6.2.1 Important mathematical remark.

Two finite vector spaces having the same dimension are isomorphic. Indeed, if we are given V and W with no additional linear maps either between them or linking them to another vector space, then they are, for all purposes, essentially the same.

However, if V and W have the same dimension and are both subspaces of U, they are indeed isomorphic as vector spaces, but they need not be isomorphic as subspaces of $\mathbb{U}$. That is because we cannot find an invertible linear map $f : V \to W$, such that $i_w f = i_v$ where $i_v : V \to U$ and $i_w : W \to U$ are the inclusion maps. Indeed, in the case of subspaces, there is a means to distinguish them using the inclusion maps into U. A choice of a basis for U, for instance, would helps us in doing so. Such a remark also extends to the case of module. Most of the equalities in the chapter are isomorphisms, and we should be careful as to what they imply.

6.3 Linear time-invariant systems.

We briefly review the behavioral approach to systems theory, and introduce our objects of study, the linear time-invariant systems.

6.3.1 Review of the behavioral approach.

Rather than viewing a system as an input/output device, the behavioral approach views a system as a collection of trajectories allowed possible by the laws of a model.

Definition 6.3.1 (cf. [PW98], Section 1.2.1)**.** *A Willems system is a pair* $(\mathbb{U}, \mathcal{B})$ *where* $\mathbb{U}$ *is a set, called the universum—its elements are called* outcomes*—and* $\mathcal{B}$ *a subset of* $\mathbb{U}$ *called the* behavior*.*

A system is made dynamical by considering universa of the form $\mathbb{W}^{\mathbb{T}}$, the set of maps from $\mathbb{T}$ to $\mathbb{W}$. Linearity and time-invariance emerge when the universa and the behaviors are endowed with a certain structure. We will be concerned, in this chapter, with universa of the form $\mathbb{W}^{\mathbb{T}}$ where $\mathbb{W}$ is an n-dimensional vector space over a field k, and $\mathbb{T}$ is the set of natural numbers.

6.3.2 Linear time-invariant systems.

We fix a field k throughout the chapter. The reader may wish to instantiate k to being, for instance, either $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or any finite field. We define $\mathcal{R}$ to be the ring $k[[z^{-1}]]$ of formal power series in the variable z^{-1} with coefficients in k. The elements in $\mathcal{R}$ are series of the form:

$$\sum_{i=0}^{\infty} a_i z^{-i}, \qquad \text{with } a_i \in k. \tag{6.1}$$

The series in (6.1) ought to be interpreted as a discrete-time signal taking value a_i in k at time i. Addition in $\mathcal{R}$ is given by pointwise addition:

$$\sum_i a_i z^{-i} + \sum_i b_i z^{-i} = \sum_i (a_i + b_i) z^{-i},$$

and multiplication is given by convolution:

$$(\sum_i a_i z^{-i})(\sum_i b_i z^{-i}) = \sum_i (a_0 b_i + \cdots + a_i b_0) z^{-i}.$$

Definition 6.3.2. *A linear time-invariant system (abreviated LTI system) is pair* $(\mathbb{U}, \mathcal{B})$ *where* $\mathbb{U}$ *is the free* $\mathcal{R}$*-module* $\mathcal{R}^n$ *of dimension* n*, and* $\mathcal{B}$ *is an* $\mathcal{R}$*-submodule of* $\mathbb{U}$*.*

The system $(\mathbb{U}, \mathcal{B})$ is *linear* as $\alpha s + \alpha' s' \in \mathcal{B}$ for every $\alpha, \alpha' \in k$ and $s, s' \in \mathcal{B}$. The system $(\mathbb{U}, \mathcal{B})$ is *time-invariant* as $z^{-1}s \in \mathcal{B}$ for every $s \in \mathcal{B}$.

Proposition 6.3.3. *The system* $(\mathbb{U}, \mathcal{B})$ *is a linear time-invariant system if, and only if,* $\mathbb{U} = \mathcal{R}^n$ *and* $\mathcal{B} = \ker(f)$ *where* $f : \mathbb{U} \to E$ *is a linear map for some* $\mathcal{R}$*-module* E*.*

Proof. Every injective map is the kernel of some map, e.g. its cokernel. □

Every system then admits at least one kernel representation. Of course, different maps may yield the same system.

Definition 6.3.4. *Let* $(\mathbb{U}, \mathcal{B})$ *and* $(\mathbb{U}, \mathcal{B}')$ *be LTI systems with kernel representation:*

$$\mathcal{B} = \ker(f : \mathbb{U} \to E) \quad \text{and} \quad \mathcal{B}' = \ker(f' : \mathbb{U} \to E').$$

Their interconnection is the LTI system $(\mathbb{U}, \mathcal{B} \cap \mathcal{B}')$ *with kernel representation:*

$$(f, f') : \mathbb{U} \to E \oplus E'.$$

It can be the case that E is of the form $\mathcal{R}^m$. (This is, however, not always possible.) As $\mathbb{U}$ and E are free $\mathcal{R}$-module of finite dimension, each admits a finite basis. We may then fix a basis for each, and represent the map f through an $n \times m$ matrix with coefficients in $\mathcal{R}$.

Example. *Recall our running example:*

$$\text{(A)} \quad x[n] = y[n] + y[n-1] \quad \text{and} \quad \text{(B)} \quad y[n] = x[n].$$

The universum $\mathbb{U}$ *is* $\mathcal{R}^2$*, and the systems* $(\mathbb{U}, \mathcal{B}_A)$ *and* $(\mathbb{U}, \mathcal{B}_B)$ *have, respectively, matrix representations:*

$$A := \begin{bmatrix} 1 & -1 - z^{-1} \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 1 & -1 \end{bmatrix}. \tag{6.2}$$

The behaviors $\mathcal{B}_A$ *and* $\mathcal{B}_B$ *are the submodules of* $\mathbb{U}$ *generated by* $(1 + z^{-1}, 1)$ *and* $(1, 1)$*, respectively.*

We begin by considering only systems that live in the same universum, and generalize to systems living in different universa near the end of the chapter.

6.3.3 Memoryless LTI systems.

A memoryless system is a system where time does not affect, or restrict, the trajectories. The allowable trajectories can be shifted back in time and can be patched freely together.

Definition 6.3.5. *An LTI system* $(\mathbb{U}, \mathcal{B})$ *is said to be memoryless when:*

i. if $z^{-1}s \in \mathcal{B}$ *then* $s \in \mathcal{B}$.

ii. if $\sum_{i=0}^{\infty} a_i z^{-i} \in \mathcal{B}$, *then* $\sum_{i=0}^{T} a_i z^{-i} \in \mathcal{B}$ *for all* T.

The memoryless system $(\mathbb{U}, \mathcal{B}_{mem})$ derived from $(\mathbb{U}, \mathcal{B})$ is the memoryless system we get by destroying from $(\mathbb{U}, \mathcal{B})$ any capacity it has to remember the past.

Definition+Proposition 6.3.6. *The behavior of the memoryless system* $(\mathbb{U}, \mathcal{B}_{mem})$ *derived from an LTI system* $(\mathbb{U}, \mathcal{B})$ *is the* $\mathcal{R}$*-submodule* $\mathcal{B}_{mem}$ *of* $\mathbb{U}$ *generated by the signals* $a_0 \in k^n \subset \mathbb{U}$ *where* $a_0 + z^{-1}s \in \mathcal{B}$ *for some* $s \in \mathbb{U}$.

Notice that a behavior $\mathcal{B}_{mem}$ when $\mathbb{U} = \mathcal{R}^n$, in addition to having a $\mathcal{R}$-module structure, can be regarded as a k-vector subspace of k^n.

Example. *The systems in the running example have as* $(\mathcal{B}_A)_{mem} = (\mathcal{B}_B)_{mem}$ *the* $\mathcal{R}$*-module spanned by* $(1, 1)$, *or equivalently the subspace of* k^2 *spanned by* $(1, 1)$.

6.3.4 Some theorems on $\mathcal{R}$-modules.

We will be only concerned with modules over $\mathcal{R} = k[[z^{-1}]]$ in this chapter. Most of the results throughout the chapter generally hold for rings that are principal ideal domains, such as the ring $\mathbb{Z}$ of integers, and the polynomial ring $k[x]$. A ring is said to be a principal ideal domain if the product of two non-zero elements is non-zero, and every ideal is generated by a single element.

Recall that a module is said to be free if it is a direct sum of copies of $\mathcal{R}$.

Proposition 6.3.7. *Let* M *be a free* $\mathcal{R}$*-module. If* N *is a submodule of* M, *then* N *is free. If* $\dim(N)$ *denotes the dimension of* N, *then* $\dim(N) \leq \dim(M)$.

Proof. See e.g. [Lan02] Ch. III Theorem 7.1. □

An LTI system $(\mathbb{U}, \mathcal{B})$ is then composed of two free $\mathcal{R}$-module. To capture the role of memory, we will need *torsion* in our modules. As such, we will mostly be working with the module $\mathbb{U}/\mathcal{B}$ derived from a system $(\mathbb{U}, \mathcal{B})$. Such modules are very general:

Proposition 6.3.8. *Every finitely generated* $\mathcal{R}$*-module is of the form* $\mathbb{U}/\mathcal{B}$ *for some free module* $\mathbb{U}$ *of finite dimension and a submodule* $\mathcal{B}$.

Proof. An $\mathcal{R}$-module module M is finitely generated if it has a finite number n of generators. There then exists a surjective linear map $\mathcal{R}^n \to M$. The module M is then the cokernel of the kernel $\mathcal{B}$ of the map f, namely of the form $\mathcal{R}^n/\mathcal{B}$. □

The following characterization will be useful in clarifying the ideas.

Proposition 6.3.9. *Let M be a finitely generated $\mathcal{R}$-module, then there exists an integer m and elements $s_0, \cdots, s_l \in \mathcal{R}$ such that:*

$$M = \mathcal{R}^m \oplus \mathcal{R}/(s_0\mathcal{R}) \oplus \cdots \oplus \mathcal{R}/(s_l\mathcal{R}).$$

The elements $s_0, \cdots, s_l$ can be chosen so that they are powers of primes in $\mathcal{R}$.

Proof. See e.g., [Lan02] Ch. III Theorem 7.3 and Theorem 7.5, combined. □

The submodule $\mathcal{R}^m$ is known as the free submodule of M, while $\mathcal{R}/(s_0\mathcal{R}) \oplus \cdots \oplus \mathcal{R}/(s_l\mathcal{R})$ is known as the torsion submodule of M.

Example 1. *For instance, both $\mathbb{U}/\mathcal{B}_A$ and $\mathbb{U}/\mathcal{B}_B$, in our running example, are isomorphic to $\mathcal{R}$.*

Note that a module may admit multiple factorizations of the above form, as for instance:

$$\mathcal{R}/((z^{-2} + z^{-1})\mathcal{R}) = \mathcal{R}/(z^{-1}\mathcal{R}) \oplus \mathcal{R}/((z^{-1} + 1)\mathcal{R})$$

The proposition 6.3.9 can be further strengthened, but will be enough for our purposes. As we shall see, the dimension of $\mathcal{B}_{mem}$ derived from $(\mathbb{U}, \mathcal{B})$ will be $n - m - d$ where m is the dimension of the free submodule of $\mathbb{U}/\mathcal{B}$, and d is dimension the largest submodule of $\mathbb{U}/\mathcal{B}$ isomorphic to a direct sum of modules of the form $\mathcal{R}/(z^{-i}\mathcal{R})$.

6.4 The role of memory.

Given two systems $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$, our experiment tells us that the role played by memory during their interaction appears through the inequality:

$$(\mathcal{B} \cap \mathcal{B}')_{mem} \neq \mathcal{B}_{mem} \cap \mathcal{B}'_{mem}.$$

Example 2. *For instance, our running example has $(\mathcal{B}_A)_{mem} \cap (\mathcal{B}_B)_{mem}$ as the subspace of k^2 generated by $(1, 1)$, however $(\mathcal{B}_A \cap \mathcal{B}_B)_{mem}$ is the 0 vector space.*

The goal is to quantify the inequality and cope with it. How can we non-trivially relate $(\mathcal{B} \cap \mathcal{B}')_{mem}$ to $\mathcal{B}$ and $\mathcal{B}'$? We can then characterize the role played by memory, and understand its effects.

6.4.1 Interconnection and exact sequences

The notion of exact sequence is crucial. Interconnection of systems can be first expressed as an exact sequence. Loss of memory will then cause a loss of exactness. Finally, the memoryless interconnected system will be related to its separate component systems through exact sequences.

Definition 6.4.1. *A sequence of $\mathcal{R}$-modules M_i and $\mathcal{R}$-modules homomorphisms f_i*

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

is said to be exact at M_i if $\operatorname{im} f_i = \ker f_{i+1}$. *The sequence is called an exact sequence if it is exact at every M_i.*

Let $N \subseteq N'$ be submodules of M. The inclusion map $f : N \to N'$ induces a linear map $g : M/N \to M/N'$. The map g is surjective as f is injective. In particular, if $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ are LTI systems, we then obtain four canonical surjective maps:

$$\begin{aligned} p &: \mathbb{U}/(\mathcal{B} \cap \mathcal{B}') \to \mathbb{U}/\mathcal{B} \qquad & p' &: \mathbb{U}/(\mathcal{B} \cap \mathcal{B}') \to \mathbb{U}/\mathcal{B}' \\ q &: \mathbb{U}/\mathcal{B} \to \mathbb{U}/(\mathcal{B} + \mathcal{B}') \qquad & q' &: \mathbb{U}/\mathcal{B}' \to \mathbb{U}/(\mathcal{B} + \mathcal{B}') \end{aligned}$$

Proposition 6.4.2. *If $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ are LTI systems, then the sequence:*

$$0 \longrightarrow \mathbb{U}/(\mathcal{B} \cap \mathcal{B}') \xrightarrow{(p,p')} \mathbb{U}/\mathcal{B} \oplus \mathbb{U}/\mathcal{B}' \xrightarrow{q-q'} \mathbb{U}/(\mathcal{B} + \mathcal{B}') \longrightarrow 0$$

is exact.

Proof. Construct a commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{B} \cap \mathcal{B}' \arrow[r] \arrow[d] & \mathcal{B} \oplus \mathcal{B}' \arrow[r] \arrow[d] & \mathcal{B} + \mathcal{B}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{U} \arrow[r] \arrow[d] & \mathbb{U} \oplus \mathbb{U} \arrow[r] \arrow[d] & \mathbb{U} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{U}/(\mathcal{B} \cap \mathcal{B}') \arrow[r] \arrow[d] & \mathbb{U}/\mathcal{B} \oplus \mathbb{U}'/\mathcal{B}' \arrow[r] \arrow[d] & \mathbb{U}/(\mathcal{B} + \mathcal{B}') \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The top two rows are exact. Either apply the nine-lemma (or the 3×3 lemma, see e.g. [Wei95] Ch. 1 Exercise 1.3.2) to get that the bottom row is exact, apply the Snake lemma (see later Proposition 6.4.6) on the top two rows. □

To clarify, the linear map $q - q'$ sends $(s, s') \in \mathbb{U}/\mathcal{B} \oplus \mathbb{U}/\mathcal{B}'$ to $q(s) - q'(s') \in \mathbb{U}/(\mathcal{B} + \mathcal{B}')$. Thus to be notationally precise, the map $q - q'$ should be denoted as $q\pi - q'\pi'$ for some projections π and π'.

Example 3. *If $\mathbb{U} = \mathcal{R}^n$ and $S \subset \mathbb{U}$, then we denote by $\langle S \rangle$ the $\mathcal{R}$-submodule of $\mathbb{U}$ generated by the elements in S. The exact sequence, resulting from our running example, can be explicitly expressed as:*

$$0 \longrightarrow \mathbb{U}/\langle 0 \rangle \xrightarrow{(0,0)} \mathbb{U}/\langle (1,1) \rangle \oplus \mathbb{U}/\langle (1, 1 + z^{-1}) \rangle \xrightarrow{q-q'} \mathbb{U}/\langle (1,1), (0, z^{-1}) \rangle \longrightarrow 0$$

Note that $\{(1,1), (1, 1 + z^{-1})\}$ *and* $\{(1,1), (0, z^{-1})\}$ *generate the same submodule.*

Of course, the sequence:

$$0 \longrightarrow \mathcal{B} \cap \mathcal{B}' \longrightarrow \mathcal{B} \oplus \mathcal{B}' \longrightarrow \mathcal{B} + \mathcal{B}' \longrightarrow 0$$

is also exact. The main issue is that the modules in that sequence are always free modules. We will need to consider instead the exact sequence of quotients, generally not consisting of free modules. As we mentioned, we will need to use the *torsion* that comes from the quotients to capture the loss of memory.

6.4.2 Loss of memory and loss of exactness

Let $(\mathbb{U}, \mathcal{B})$ be an LTI system. Recall that $(k^n, \mathcal{B}_{mem})$ (or equivalently $(\mathbb{U}, \mathcal{B}_{mem})$) is the memoryless system derived from $(\mathbb{U}, \mathcal{B})$. Precisely, we have:

$$\mathcal{B}_{mem} := \{s_0 \in k^n : s_0 + z^{-1}s \in \mathcal{B} \text{ for some } s \in \mathbb{U}\}.$$

Let M be an $\mathcal{R}$-module, we define the $\mathcal{R}$-module:

$$\Phi M := M = M/(z^{-1}M).$$

The elements of ΦM are the equivalence classes of M, where s and s' belong to the same equivalence class if, and only if, $s - s' \in z^{-1}M$. The module ΦM is obtained by tensoring M with $\mathcal{R}/z^{-1}\mathcal{R}$. We briefly discuss this connection later on in the chapter.

Proposition 6.4.3. *If $(\mathbb{U}, \mathcal{B})$ is an LTI system, then $\Phi(\mathbb{U}/\mathcal{B}) = k^n/\mathcal{B}_{mem}$.*

Proof. We have:

$$\begin{aligned}
\Phi(\mathbb{U}/\mathcal{B}) &= (\mathbb{U}/\mathcal{B})/z^{-1}(\mathbb{U}/\mathcal{B}) \quad \text{(by definition)}\\
&= (\mathbb{U}/\mathcal{B})/\big((\mathcal{B} + z^{-1}\mathbb{U})/\mathcal{B}\big)\\
&= \mathbb{U}/(\mathcal{B} + z^{-1}\mathbb{U}) \quad \text{(by the third isomorphism theorem)}\\
&= k^n/\mathcal{B}_{mem}.
\end{aligned}$$

The second equality (isomorphism) follows from $z^{-1}(u+\mathcal{B}) = z^{-1}u+\mathcal{B} \in (\mathcal{B}+z^{-1}\mathbb{U})/\mathcal{B}$ whenever $u + \mathcal{B} \in \mathbb{U}/\mathcal{B}$. (See e.g. [AM69] Ch. 2 p. 18 for more information, and the proof of Corollary 2.7 for a use of this fact.) □

As both k^n and $\mathcal{B}_{mem}$ are k-vector spaces, we have:

$$k^n = \mathcal{B}_{mem} \oplus k^n/\mathcal{B}_{mem}.$$

Thus, characterizing $\mathcal{B}_{mem}$ is equivalent to characterizing $k^n/\mathcal{B}_{mem}$.

The operation Φ may also be lifted to linear maps. Indeed, let $f : M \to N$ be a linear map, then we may define $\Phi(f) : \Phi(M) \to \Phi(N)$ to be the map:

$$\Phi f : s + z^{-1}M \longmapsto f(s) + z^{-1}N \qquad \text{for } s \in M$$

Given an exact sequence, we may then apply Φ to both the objects and the morphisms to get another sequence. The key point of the chapter is that the act of destroying memory does not behave well under interconnection, and will cause a loss of exactness. Indeed:

Proposition 6.4.4. *If $0 \to M \to M' \to M'' \to 0$ be an exact sequence of $\mathcal{R}$-modules, then the sequence $0 \to \Phi M \to \Phi M' \to \Phi M'' \to 0$ is always exact at $\Phi M'$ and $\Phi M''$.*

Proof. Notice that $\Phi M = M \otimes \mathcal{R}/z^{-1}\mathcal{R}$, and see e.g., [AM69] Ch 2 Proposition 2.18. □

However:

Proposition 6.4.5. *The map Φ does not always send injective linear maps to injective linear maps. In particular, the sequence $0 \to \Phi M \to \Phi M' \to \Phi M'' \to 0$ is not always exact at ΦM.*

Proof. See e.g., the example below. □

Exactness is lost on the left precisely when memory plays a role in our experiment.

Example. *Returing to our running example, the sequence:*

$$0 \to \Phi\mathbb{U}/(\mathcal{B}_A \cap \mathcal{B}_B) \to k \oplus k \to k \to 0$$

cannot be exact at $\Phi\mathbb{U}/(\mathcal{B}_A \cap \mathcal{B}_B)$, as that would imply $\Phi\mathbb{U}/(\mathcal{B}_A \cap \mathcal{B}_B)$ to be k. We know however that $\Phi\mathbb{U}/(\mathcal{B}_A \cap \mathcal{B}_B)$, is not k but is isomorphic to k^2. Indeed, $(\mathcal{B}_A \cap \mathcal{B}_B)_{mem} = 0$.

Our goal is to figure out a way to recover the loss.

6.4.3 Recovering exactness

We will recover the loss of exactness through the use of the snake lemma:

Proposition 6.4.6 (Snake Lemma, e.g., [AM69] ch. 2, p. 22, proposition 2.10)**.** *Given a commutative diagram of $\mathcal{R}$-modules with exact rows,*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "m"] & N \arrow[r, "g"] \arrow[d, "n"] & P \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f'"] & N' \arrow[r, "g'"] & P' \arrow[r] & 0
\end{tikzcd}$$

we get an exact sequence:

$$0 \longrightarrow \ker m \xrightarrow{\tilde{f}} \ker n \xrightarrow{\tilde{g}} \ker p \xrightarrow{\delta} \operatorname{coker} m \xrightarrow{\bar{f'}} \operatorname{coker} n \xrightarrow{\bar{g'}} \operatorname{coker} p \longrightarrow 0.$$

Proof. The lemma is standard, and its proof may be found in many texts, e.g., [AM69] ch. 2, proposition 2.10. □

The snake lemma enables us to relate cokernels of maps to their kernels through a long exact sequence. When the objects in an exact sequence are vector spaces, we may directly further express elements of it through others.

Proposition 6.4.7. *If the sequence of vector spaces:*

$$0 \longrightarrow V_0 \longrightarrow V_1 \xrightarrow{f} V_2 \longrightarrow V_3 \longrightarrow V_4 \xrightarrow{g} V_5 \longrightarrow 0,$$

is exact, then $V_0 = \ker f$ *and* $V_3 = \operatorname{coker} f \oplus \ker g$.

Proof. The following sequence is exact:

$$0 \to \operatorname{im}(V_1 \to V_2) \to V_2 \to \operatorname{im}(V_2 \to V_3) \to 0$$

We have $\operatorname{im}(V_1 \to V_2) = V_1/\ker(V_1 \to V_2)$. As $\ker(V_1 \to V_2) = \operatorname{im} f$ by exactness of the six-term sequence, we get that $\operatorname{im}(V_1 \to V_2) = \operatorname{coker}(f)$. By exactness, we also get $\operatorname{im}(V_2 \to V_3) = \ker g$. Finally, short exact sequence of vectors spaces split. Namely, if $0 \to U \to V \to W \to 0$ is a sequence of vector spaces, then $V = U \oplus W$. □

To make use of lemmas, notice that:

$$\Phi M = \operatorname{coker}(M \xrightarrow{z^{-1}} M).$$

We can then construct a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{U}/(\mathcal{B} \cap \mathcal{B}') \arrow[r, "{(p,p')}"] \arrow[d, "z^{-1}"] & \mathbb{U}/\mathcal{B} \oplus \mathbb{U}/\mathcal{B}' \arrow[r, "q-q'"] \arrow[d, "z^{-1}"] & \mathbb{U}/(\mathcal{B} + \mathcal{B}') \arrow[r] \arrow[d, "z^{-1}"] & 0 \\
0 \arrow[r] & \mathbb{U}/(\mathcal{B} \cap \mathcal{B}') \arrow[r, "{(p,p')}"] & \mathbb{U}/\mathcal{B} \oplus \mathbb{U}/\mathcal{B}' \arrow[r, "q-q'"] & \mathbb{U}/(\mathcal{B} + \mathcal{B}') \arrow[r] & 0
\end{tikzcd}$$

Let M be an $\mathcal{R}$-module, we define the $\mathcal{R}$-module:

$$\mathbb{H}M := \ker(M \xrightarrow{z^{-1}} M).$$

The operator $\mathbb{H}$ can also be lifted to linear maps. Indeed, let $f : M \to N$ be a linear map, we define $\mathbb{H}(f) : \mathbb{H}(M) \to \mathbb{H}(N)$ to be the restriction of f to $\mathbb{H}(M)$.

Proposition 6.4.8. *If* $(\mathbb{U}, \mathcal{B})$ *and* $(\mathbb{U}, \mathcal{B}')$ *are LTI systems, we get an exact sequence:*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{H}(\mathbb{U}/(\mathcal{B} \cap \mathcal{B}')) \arrow[r, "{(\mathbb{H}(p),\mathbb{H}(p'))}"] & \mathbb{H}(\mathbb{U}/\mathcal{B}) \oplus \mathbb{H}(\mathbb{U}/\mathcal{B}') \arrow[r, "\mathbb{H}(q)-\mathbb{H}(q')"] & \mathbb{H}(\mathbb{U}/(\mathcal{B} + \mathcal{B}')) \arrow[dlll] \\
\Phi(\mathbb{U}/(\mathcal{B} \cap \mathcal{B}')) \arrow[r, "{(\Phi(p),\Phi(p'))}"'] & \Phi(\mathbb{U}/\mathcal{B}) \oplus \Phi(\mathbb{U}/\mathcal{B}') \arrow[r, "\Phi(q)-\Phi(q')"'] & \Phi(\mathbb{U}/(\mathcal{B} + \mathcal{B}')) \arrow[r] & 0
\end{tikzcd}$$

Proof. Apply the Snake lemma (Proposition 6.4.6) to the obtained ladder diagram. □

Furthermore:

Theorem 6.4.9. *We have:*

$$\Phi(\mathbb{U}/(\mathcal{B}\cap\mathcal{B}')) = \ker(\Phi(q)-\Phi(q'))\oplus\operatorname{coker}(\mathbb{H}(q)-\mathbb{H}(q'))$$

and:

$$\mathbb{H}(\mathbb{U}/(\mathcal{B}\cap\mathcal{B}')) = \ker(\mathbb{H}(q)-\mathbb{H}(q'))$$

Proof. Combine Proposition 6.4.8 and Proposition 6.4.7. □

The space $\ker(\Phi(q)-\Phi(q'))$ reflects what comes from the interaction of the memoryless systems. The space $\operatorname{coker}(\mathbb{H}(q)-\mathbb{H}(q'))$, on the other hand, captures the effect of memory on the interconnected memoryless system.

Example 4. *In our running example, both $\mathbb{H}(\mathbb{U}/\mathcal{B}_A)$ and $\mathbb{H}(\mathbb{U}/\mathcal{B}_B)$ will be the 0 vector spaces, and both $\mathbb{H}(q)$ and $\mathbb{H}(q')$ will be the 0 maps. We then get:*

$$\operatorname{coker}(\mathbb{H}(q)-\mathbb{H}(q')) = \mathbb{H}(\mathbb{U}/(\mathcal{B}_A+\mathcal{B}_B)) \quad \textit{and} \quad \ker(\mathbb{H}(q)-\mathbb{H}(q'))=0$$

6.5 Interpreting the recovery.

Theorem 6.4.9 in the previous section leaves us with three expressions:

$$\ker(\Phi(q)-\Phi(q')) \qquad \operatorname{coker}(\mathbb{H}(q)-\mathbb{H}(q')) \qquad \ker(\mathbb{H}(q)-\mathbb{H}(q'))$$

to thoroughly interpret. We interpret them, and proceed to elucidate the operator $\mathbb{H}$.

Definition 6.5.1. *Let N be a submodule of M, we denote by zN the module:*

$$zN := \{s\in M : z^{-1}s\in N\}.$$

The notation zN does not explicitly refer to M, however, in all its usage, the module M will be clear from the context.

Remark. The element $z^{-1}\in\mathcal{R}$ does not have an inverse in $\mathcal{R}$. The letter z is undefined and is not an element of $\mathcal{R}$. In other words, we cannot multiply signals by z in the ring. The notation zN is intended to divide signals in N by z^{-1} only if possible, as if we fictively multiplied them by z.

6.5.1 Elements of the long exact sequence.

Our experiment tells us that memory plays a role in the interaction, when:

$$(\mathcal{B}\cap\mathcal{B}')_{mem}\neq\mathcal{B}_{mem}\cap\mathcal{B}'_{mem}.$$

First, the interaction of the memoryless systems is captured through:

Proposition 6.5.2. *We have:*

$$\ker(\Phi(q) - \Phi(q')) = k^n/(\mathcal{B}_{mem} \cap \mathcal{B}'_{mem}).$$

Proof. The map $\Phi(q)$ is the canonical map $k^n/\mathcal{B}_{mem} \to k^n/(\mathcal{B}_{mem} + \mathcal{B}'_{mem})$. A similar fact holds for $\Phi(q')$. The sequence:

$$0 \to k^n/(\mathcal{B}_{mem} \cap \mathcal{B}'_{mem}) \to k^n/\mathcal{B}_{mem} \oplus k^n/\mathcal{B}'_{mem} \xrightarrow{\Phi(q)-\Phi(q')} k^n/(\mathcal{B}_{mem} + \mathcal{B}'_{mem}) \to 0$$

is then exact. □

Recall that $\mathbb{H}(M) = \ker(M \xrightarrow{z^{-1}} M)$, then:

Proposition 6.5.3. *We have* $\mathbb{H}(\mathbb{U}/\mathcal{B}) = z\mathcal{B}/\mathcal{B}$ *where* $z\mathcal{B} = \{s \in \mathbb{U} : z^{-1}s \in \mathcal{B}\}$.

Proof. If $s + \mathcal{B} \in \mathbb{U}/\mathcal{B}$, we then have: $z^{-1}(s + \mathcal{B}) = \mathcal{B}$ if, and only if, $z^{-1}s \in \mathcal{B}$. Of course, $z^{-1}(s + \mathcal{B}) = \mathcal{B}$ if, and only if, $s + \mathcal{B} \in \mathbb{H}(\mathbb{U}/\mathcal{B})$. □

The role of (the destroyed) memory in the interaction is then captured through:

Proposition 6.5.4. *We have:*

$$\operatorname{coker}(\mathbb{H}(q) - \mathbb{H}(q')) = z(\mathcal{B} + \mathcal{B}')/(z\mathcal{B} + z\mathcal{B}').$$

Proof. We have $\operatorname{im}(\mathbb{H}(q) - \mathbb{H}(q')) = (z\mathcal{B} + \mathcal{B}')/(\mathcal{B} + \mathcal{B}') + (\mathcal{B} + z\mathcal{B}')/(\mathcal{B} + \mathcal{B}') = (z\mathcal{B} + z\mathcal{B}')/(\mathcal{B} + \mathcal{B}')$. The rest follows by the third isomorphism theorem. □

The space $z^{-1}(\mathcal{B} + \mathcal{B}')/(z^{-1}\mathcal{B} + z^{-1}\mathcal{B}')$ is a k-vector space isomorphic to:

$$\begin{aligned} I := \{0\} \cup \{a \in k^n :\ & \text{there exists } (s, s') \in \mathcal{B} \oplus \mathcal{B}' \text{ with } s[0] = s'[0] = a \\ & \text{such that for every } (t, t') \in \mathcal{B} \oplus \mathcal{B}' \text{ with } t[0] = t'[0] = 0,\ s + t \neq s' + t'\}. \end{aligned}$$

when viewed as a subspace of k^n. We then get $a \in I$ if, and only if, $a \in \mathcal{B}_{mem} \cap \mathcal{B}'_{mem}$ but there exists no $s \in \mathcal{B} \cap \mathcal{B}'$ with $s[0] = a$. We can thus recover:

$$(\mathcal{B} \cap \mathcal{B}')_{mem} = (\mathcal{B}_{mem} \cap \mathcal{B}'_{mem})/I.$$

Or equivalently, we have:

$$(\mathcal{B}_{mem} \cap \mathcal{B}'_{mem})/(\mathcal{B} \cap \mathcal{B}')_{mem} = z(\mathcal{B} + \mathcal{B}')/(z\mathcal{B} + z\mathcal{B}').$$

A close look at the equation reveals a duality between $\mathcal{B}_{mem}$ and $z\mathcal{B}$ (or more precisely, $z\mathcal{B}/\mathcal{B}$).

Corollary 6.5.5. *We thus get:*

$$\mathbb{U}/(\mathcal{B} \cap \mathcal{B}')_{mem} = \mathbb{U}/(\mathcal{B}_{mem} \cap \mathcal{B}'_{mem}) \oplus z(\mathcal{B} + \mathcal{B}')/(z\mathcal{B} + z\mathcal{B}').$$

Proof. Immediate from Theorem 6.4.9. □

Finally, the role that memory could potentially play in the new interconnected system consists of:

Proposition 6.5.6. *We have:*

$$\ker(\mathbb{H}(q) - \mathbb{H}(q')) = (z\mathcal{B} \cap z\mathcal{B}')/(\mathcal{B} \cap \mathcal{B}').$$

Proof. The statement follows from $z\mathcal{B} \cap z\mathcal{B}' = z(\mathcal{B} \cap \mathcal{B}')$ and Theorem 6.4.9. □

6.5.2 The functorial nature of Φ and $\mathbb{H}$.

The operators Φ and $\mathbb{H}$ act both on $\mathcal{R}$-modules and on $\mathcal{R}$-linear maps. They send $\mathcal{R}$-modules to $\mathcal{R}$-modules and $\mathcal{R}$-linear maps to $\mathcal{R}$-linear maps. The operators Φ and $\mathbb{H}$ are usually termed *functors*, as they satisfy two properties:

i. If $f : M \to N$ and $g : N \to P$ are $\mathcal{R}$-linear maps, then $\Phi(f \circ g) = \Phi(f) \circ \Phi(g)$ and $\mathbb{H}(f \circ g) = \mathbb{H}(f) \circ \mathbb{H}(g)$

ii. If $id : M \to M$ is the identity map on M, then both $\Phi(id)$ and $\mathbb{H}(id)$ are the identity maps on $\Phi(M)$ and $\mathbb{H}(M)$.

The functor Φ consists of tensoring a module with $\mathcal{R}/z^{-1}\mathcal{R}$. Given a cyclic factorization (cf. Proposition 6.3.9) of a module M, the module $\Phi(M)$ is isomorphic to $(\mathcal{R}/(z^{-1}\mathcal{R}))^m$ where m is the number of factors of the form either $\mathcal{R}$ or $\mathcal{R}/(z^{-i}\mathcal{R})$. The functor $\mathbb{H}$ will be seen to send M to the set of linear maps $hom(\mathcal{R}/z^{-1}\mathcal{R}, M)$ from $\mathcal{R}/z^{-1}\mathcal{R}$ to M. Both of those operators can be seen to lift to linear maps.

Taking tensor products, e.g., Φ, is known to be *right-exact*, but not generally *left-exact*. In other words, Φ preserves exactness of an exact sequence on the right, but not necessarily on the left. Equivalently, Φ generally fails to send injective maps to injective maps. The fact that it fails to be left-exact is the reason why memory could potentially play a role. The functor $\mathbb{H}$ is termed the first-order left derived functor of Φ, and is interpreted in measuring by how much exactness is lost. There are principled techniques to define derived functors, for instance through projective resolutions. Those methods are outside the scope of the chapter, but in our setting, they reduce to our direct approach through the use of the snake lemma.

6.5.3 The lag of an LTI system.

The operator $\mathbb{H}$ while acting on $\mathbb{U}/\mathcal{B}$, can be singled out as an invariant—termed, *the lag*—of a linear time-invariant system.

Definition 6.5.7. *The lag of* $\operatorname{lag}(\mathbb{U}, \mathcal{B})$ *an LTI system* $(\mathbb{U}, \mathcal{B})$ *is defined to be* $\mathbb{H}(\mathbb{U}/\mathcal{B}) = z\mathcal{B}/\mathcal{B}$.

Elements in the lag correspond to classes of signals that only appears delayed in the system. If $hom(M, N)$ denotes the (abelian) group of linear maps from M to N, then:

Proposition 6.5.8. *We have* $\mathbb{H}(\mathbb{U}/\mathcal{B}) = hom(\mathcal{R}/(z^{-1}\mathcal{R}), \mathbb{U}/\mathcal{B})$.

Proof. A linear map from $\mathcal{R}/(z^{-1}\mathcal{R}) \simeq k$ to $\mathbb{U}/\mathcal{B}$ is an element s of $\mathbb{U}/\mathcal{B}$ such that $z^{-1}s = 0$ in $\mathbb{U}/\mathcal{B}$. □

The space $\mathbb{H}(M)$ is a submodule of M. The operator $\mathbb{H}$ maps $\mathcal{R}/s\mathcal{R}$ to its unique submodule $\mathcal{R}/z^{-1}\mathcal{R}$ if $s \in z^{-1}\mathbb{U}$ and to 0 otherwise. Furthermore, $\mathbb{H}$ commutes with direct sums, i.e., $\mathbb{H}(M \oplus N) = \mathbb{H}(M) \oplus \mathbb{H}(N)$. Thus, given a presentation of a module $\mathbb{U}/\mathcal{B}$, the functor $\mathbb{H}$ keeps one $\mathcal{R}/z^{-1}\mathcal{R}$ for every factor in the form $\mathcal{R}/s\mathcal{R}$ where $s \in z^{-1}\mathbb{U}$.

Proposition 6.5.9. *If $(\mathbb{U}, \mathcal{B})$ is an LTI system with $\mathbb{U} = \mathcal{R}^n$, then $\mathbb{H}(\mathbb{U}/\mathcal{B})$ is a finite dimensional k-vector space with dimension at most n.*

Proof. The module $\mathbb{U}/\mathcal{B}$ has at most n generators, thus every submodule cannot have more than n generators. Furthermore, $\mathbb{H}(\mathbb{U}/\mathcal{B})$ admits a $\mathcal{R}/z^{-1}\mathcal{R}$-module structure, and $\mathcal{R}/z^{-1}\mathcal{R}$ is a field (isomorphic to k). □

As the operator $\mathbb{H}$ extends to linear maps, we should also expect the lag to extend to transformations, or morphisms, of systems, which we next define. If $(\mathbb{U}, \mathcal{B})$ is an LTI system, we may replace the inclusion $\mathcal{B} \subseteq \mathbb{U}$ by an inclusion map $\mathcal{B} \to \mathbb{U}$.

Definition 6.5.10. *A morphism $\boldsymbol{h}$ from $(\mathbb{U}, \mathcal{B})$ to $(\mathbb{U}', \mathcal{B}')$ is defined to be a pair of maps $h_U : \mathbb{U} \to \mathbb{U}'$ and $h_B : \mathcal{B} \to \mathcal{B}'$ such that the following diagram commutes:*

$$\begin{tikzcd}
B \arrow[r, "\subseteq"] \arrow[d, "\phi_B"] & U \arrow[d, "\phi_U"] \\
B' \arrow[r, "\subseteq"] & U'
\end{tikzcd}$$

Given a morphism $\mathbf{h}$ we can then form a map $\mathbb{U}/\mathcal{B} \to \mathbb{U}'/\mathcal{B}'$, by first forming the map $\mathbb{U} \to \mathbb{U}'/\mathcal{B}'$ then factoring it through $\mathbb{U} \to \mathbb{U}/\mathcal{B}$ as a consequence of the commuting diagram. Conversely, linear maps of the form $\mathbb{U}/\mathcal{B} \to \mathbb{U}'/\mathcal{B}'$ arise from morphisms of systems.

Definition 6.5.11. *Let $\boldsymbol{h}$ be a morphism from $(\mathbb{U}, \mathcal{B})$ to $(\mathbb{U}', \mathcal{B}')$, the lag $\mathrm{lag}(h)$ of $\boldsymbol{h}$ is the k-linear map $\mathbb{H}(q) : \mathbb{H}(\mathbb{U}/\mathcal{B}) \to \mathbb{H}(\mathbb{U}'/\mathcal{B}')$ where $q : \mathbb{U}/\mathcal{B} \to \mathbb{U}/\mathcal{B}$ is the map induced by $\mathbb{H}$.*

Given a linear map $q : \mathbb{U}/\mathcal{B} \to \mathbb{U}'/\mathcal{B}'$, we may construct the map $\mathbb{H}(q)$ by restricting q to $\mathbb{H}(\mathbb{U}/\mathcal{B})$. Given a morphism $\mathbf{h} : (\mathbb{U}, \mathcal{B}) \to (\mathbb{U}', \mathcal{B}')$, we can denote by $\mathbf{h}(\mathbb{U}, \mathcal{B})$ the system $(h_U\mathbb{U}, h_B\mathcal{B})$. The main theorem can then be restated as follows:

Theorem 6.5.12. *Let $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ be LTI systems, and let $\boldsymbol{h} : (\mathbb{U}, \mathcal{B}) \to (\mathbb{U}, \mathcal{B} + \mathcal{B}')$ and $\boldsymbol{h}' : (\mathbb{U}, \mathcal{B}') \to (\mathbb{U}, \mathcal{B} + \mathcal{B}')$ be the canonical morphisms, then:*

$$\begin{aligned}
\mathcal{B}_{mem} \cap \mathcal{B}'_{mem}/(\mathcal{B} \cap \mathcal{B}')_{mem} &= \mathrm{coker}(\mathrm{lag}\,\boldsymbol{h} - \mathrm{lag}\,\boldsymbol{h}') \\
&= \mathrm{lag}(\mathbb{U}, \mathcal{B} + \mathcal{B}')/(\mathrm{lag}\,\boldsymbol{h}\,\mathrm{lag}(\mathbb{U}, \mathcal{B}) + \mathrm{lag}\,\boldsymbol{h}'\,\mathrm{lag}(\mathbb{U}, \mathcal{B}'))
\end{aligned}$$

Proof. The statement is immediate from 6.4.9 and the definitions in this subsection. □

We now turn to more properties and consequences.

6.6 Properties and derived consequences.

The operator $\mathbb{H}$ extracts information from the LTI systems (and the morphisms relating them) needed to characterize the role played by memory during interaction.

6.6.1 The operator $\mathbb{H}$ destroys information.

The operator $\mathbb{H}$ is keeping at least what is necessary to characterize the role of memory. The operator $\mathbb{H}$ however destroys information we do not need from the systems for understanding the role of memory.

Proposition 6.6.1. *Let $f : M \to N$ be a $\mathcal{R}$-linear map. If $\mathbb{H}(f)$ is an isomorphism, then f does not have to be an isomorphism.*

Proof. As a trivial example, consider the canonical inclusion:

$$\mathcal{R}/z^{-1}\mathcal{R} \longrightarrow \mathcal{R}/z^{-1}\mathcal{R} \oplus \mathcal{R}/(1+z^{-1})\mathcal{R}.$$

□

Furthermore, both Φ and $\mathbb{H}$ combined are not enough to recover a system.

Proposition 6.6.2. *Let $f : M \to N$ be a $\mathcal{R}$-linear map. If $\mathbb{H}(f)$ and $\Phi(f)$ are isomorphisms, then f does not have to be an isomorphism.*

Proof. The same example holds. Indeed, consider the canonical inclusion:

$$\mathcal{R}/z^{-1}\mathcal{R} \longrightarrow \mathcal{R}/z^{-1}\mathcal{R} \oplus \mathcal{R}/(1+z^{-1})\mathcal{R}.$$

□

This loss of information is not only on the level of a single system, but rather on the whole interaction situation. Given a commutative diagram of $\mathcal{R}$-modules with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "m"] & N \arrow[r, "g"] \arrow[d, "n"] & P \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f'"] & N' \arrow[r, "g'"] & P' \arrow[r] & 0
\end{tikzcd}$$

by applying Φ and $\mathbb{H}$, we get a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{H}(M) \arrow[r] \arrow[d, "\mathbb{H}(m)"'] & \mathbb{H}(N) \arrow[r] \arrow[d, "\mathbb{H}(n)"'] & \mathbb{H}(P) \arrow[r, "\delta"] \arrow[d, "\mathbb{H}(p)"'] & \Phi(M) \arrow[r] \arrow[d, "\Phi(m)"'] & \Phi(N) \arrow[r] \arrow[d, "\Phi(n)"'] & \Phi(P) \arrow[r] \arrow[d, "\Phi(p)"'] & 0 \\
0 \arrow[r] & \mathbb{H}(M') \arrow[r] & \mathbb{H}(N') \arrow[r] & \mathbb{H}(P') \arrow[r, "\delta'"] & \Phi(M') \arrow[r] & \Phi(N') \arrow[r] & \Phi(P') \arrow[r] & 0
\end{tikzcd}$$

Proposition 6.6.3. *If the linear maps $\mathbb{H}(m)$, $\mathbb{H}(n)$, $\mathbb{H}(p)$, $\Phi(m)$, $\Phi(n)$ and $\Phi(p)$ are isomorphisms, then neither m, n nor p have to be an isomorphism.*

Proof. See the example below. □

All the results hold even if we restrict the modules to be finitely generated. Such results should be intuitively hinted at as both Φ and $\mathbb{H}$ yield finite-dimensional vector spaces of dimension at most n if $\mathbb{U} = \mathcal{R}^n$.

Example. *As an example, let us consider two systems $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$, whose behaviors correspond to:*

$$\mathcal{B}: \; x[n] + x[n-1] = y[n] + 2y[n-1] \quad \text{and} \quad \mathcal{B}': \; x[n] + x[n-1] = y[n]$$

We have that $\mathbb{U}/\mathcal{B}$ and $\mathbb{U}/\mathcal{B}'$ are not isomorphic, however:

$$\mathcal{B}_{mem} = \mathcal{B}'_{mem} \text{ and } \mathbb{H}(\mathbb{U}/\mathcal{B}) = \mathbb{H}(\mathbb{U}/\mathcal{B}').$$

Thus, keeping only $\mathbb{H}(\mathbb{U}/\mathcal{B})$ and $\mathcal{B}$ cannot be enough to reconstruct $\mathcal{B}$. Furthermore, suppose we decide to interconnect each of B and B', separately, to:

$$\mathcal{B}'': \; x[n] = y[n]$$

The long exact sequences resulting from interconnecting B and B'' on one end, and B and B'' on another, are indistinguishable.

We established that $\mathbb{H}$ destroys information. We can now prove that we cannot throw away more information than what $\mathbb{H}$ keeps.

6.6.2 The operator $\mathbb{H}$ is the universal piece of information.

Recall that a functor is an operator that, in our settings, acts on $\mathcal{R}$-modules and $\mathcal{R}$-linear maps. It sends an $\mathcal{R}$-module to an $\mathcal{R}$-module and an $\mathcal{R}$-linear map to an $\mathcal{R}$-linear map. We can formally think of extracting information from our systems, or specifically from our modules, as defining a functor.

Definition 6.6.4. *A functor G is said to explain Φ if for every exact sequence $0 \to M \to N \to P \to 0$, there exists a linear map $\delta : G(P) \to \Phi(M)$ such that the following sequence:*

$$0 \to G(M) \to G(N) \to G(P) \xrightarrow{\delta} \Phi(M) \to \Phi(N) \to \Phi(P) \to 0$$

is exact.

If G explains Φ, then G can be used to characterize the role of memory during interaction. We can show that $\mathbb{H}$ is the *universal* functor that explains Φ.

Proposition 6.6.5. *Let G be a functor. If G explains Φ, then there exists a unique collection of maps $\alpha_M : G(M) \to \mathbb{H}(M)$, one for every module M, such that, for every exact sequence $0 \to M \to N \to P \to 0$, the diagram:*

$$\begin{tikzcd}
0 \arrow[r] & G(M) \arrow[r] \arrow[d, "\alpha_M"'] & G(N) \arrow[r] \arrow[d, "\alpha_N"'] & G(P) \arrow[r] \arrow[d, "\alpha_P"'] & \Phi(M) \arrow[r] \arrow[d, "\simeq"'] & \Phi(N) \arrow[r] \arrow[d, "\simeq"'] & \Phi(P) \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & \mathbb{H}(M) \arrow[r] & \mathbb{H}(N) \arrow[r] & \mathbb{H}(P) \arrow[r] & \Phi(M) \arrow[r] & \Phi(N) \arrow[r] & \Phi(P) \arrow[r] & 0
\end{tikzcd}$$

commutes.

Proof. The proof is outside the scope of the chapter, we refer to [Ada17g] for details. □

Theorem 6.6.6. *For every module M, the map $\alpha_M : G(M) \to \mathbb{H}(M)$ is surjective.*

Proof. The proof is outside the scope of the chapter, we refer to [Ada17g] for the proof. □

Thus every other information that can explain the role of memory projects onto $\mathbb{H}$. The functor $\mathbb{H}$ is, in this sense, the minimal information we can hope for.

6.6.3 Interconnection in different universa.

Before turning to more implications, we revisit our notion of interconnection and generalize it. If we are given two systems $(\mathbb{U} \oplus \mathbb{U}_c, \mathcal{B})$ and $(\mathbb{U}_c \oplus \mathbb{U}', \mathcal{B}')$, interconnecting them by sharing the universum $\mathbb{U}_c$ yields the system:

$$(\mathbb{U} \oplus \mathbb{U}_c \oplus \mathbb{U}', \mathcal{B} \oplus \mathbb{U}' \cap \mathbb{U} \oplus \mathcal{B}').$$

The interconnection is achieved by lifting the systems to a common lifted universum and then performing the intersection. We can however bypass such an explicit lift through the following observation:

$$(\mathbb{U} \oplus \mathbb{U}_c \oplus \mathbb{U}')/(\mathcal{B} \oplus \mathbb{U}' + \mathbb{U} \oplus \mathcal{B}') = \mathbb{U}_c/(\pi\mathcal{B} + \pi'\mathcal{B}'),$$

where $\pi\mathcal{B}$ and $\pi'\mathcal{B}'$ denote the projection of $\mathcal{B}$ and $\mathcal{B}'$ onto $\mathbb{U}_c$. Formally, let us define $\pi : \mathbb{U} \oplus \mathbb{U}_c \to: \mathbb{U}_c$ and $\pi' : \mathbb{U}' \oplus \mathbb{U}_c \to \mathbb{U}_c$ be the projections onto the $\mathbb{U}_c$-coordinate, i.e. $p : (u, u_c) \mapsto u_c$.

Proposition 6.6.7. *If $(\mathbb{U} \oplus \mathbb{U}_c, \mathcal{B})$ and $(\mathbb{U}' \oplus \mathbb{U}_c, \mathcal{B}')$ are LTI systems, then the behavior $\mathcal{B}^*$ of the interconnected system by sharing $\mathbb{U}_c$ is:*

$$\mathcal{B}^* = \ker(\mathcal{B} \oplus \mathcal{B}' \xrightarrow{\pi - \pi'} \mathbb{U}_c)$$

Proof. We get $\mathcal{B}^* \subseteq \mathbb{U} \oplus \mathbb{U}_c \oplus \mathbb{U}$ such that $(s, s_c, s') \in \mathcal{B}^*$ if, and only if, $(s, s_c) \in \mathcal{B}$ and $(s_c, s') \in \mathcal{B}')$. We refer the reader to [Ada17d] for more details on interconnections of systems living in different universa. □

Furthermore, if $\mathbb{U}^*$ denotes $\mathbb{U} \oplus \mathbb{U}_c \oplus \mathbb{U}'$, we get that:

Proposition 6.6.8. *The following canonical sequence:*

$$0 \longrightarrow \mathbb{U}^*/\mathcal{B}^* \longrightarrow (\mathbb{U} \oplus \mathbb{U}_c)/\mathcal{B} \oplus (\mathbb{U}' \oplus \mathbb{U}'_c)/\mathcal{B} \longrightarrow \mathbb{U}_c/(\pi\mathcal{B} + \pi'\mathcal{B}') \longrightarrow 0$$

is exact.

Proof. Construct a commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{B}^* \arrow[r] \arrow[d] & \mathcal{B} \oplus \mathcal{B}' \arrow[r] \arrow[d] & \pi(\mathcal{B} + \mathcal{B}') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{U}^* \arrow[r] \arrow[d] & (\mathbb{U} \oplus \mathbb{U}_c) \oplus (\mathbb{U}_c \oplus \mathbb{U}') \arrow[r] \arrow[d] & \mathbb{U}_c \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{U}^*/\mathcal{B}^* \arrow[r] \arrow[d] & (\mathbb{U} \oplus \mathbb{U}_c)/\mathcal{B} \oplus (\mathbb{U}_c \oplus \mathbb{U}')/\mathcal{B}' \arrow[r] \arrow[d] & \mathbb{U}_c/\pi(\mathcal{B} + \mathcal{B}') \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The top two rows are exact. Either apply the nine-lemma (or the 3×3 lemma, see e.g. [Wei95] Ch. 1 Exercise 1.3.2) to get that the bottom row is exact, apply the Snake lemma (Proposition 6.4.6) on the top two rows. □

Let us define $\iota : \mathbb{U} \oplus \mathbb{U}_c \to \mathbb{U}^*$ and $\iota' : \mathbb{U}' \oplus \mathbb{U}_c \to \mathbb{U}^*$ to be the canonical inclusions.

Corollary 6.6.9. *If $\mathbb{U}^*$ has dimension n, we get:*

$$k^n/\mathcal{B}^*_{mem} = k^n/((\iota\mathcal{B})_{mem} \cap (\iota'\mathcal{B}')_{mem}) \oplus z(\pi\mathcal{B} + \pi'\mathcal{B}')/(z\pi\mathcal{B} + z\pi'\mathcal{B}')$$

Proof. Use the same technique we used to arrive to Theorem 6.4.9. □

Thus the effects due to memory are only confined within $\mathbb{U}_c$. In case $\dim(\mathbb{U}_c) \ll \dim(\mathbb{U}^*)$, then the object $z(\pi\mathcal{B}+\pi'\mathcal{B}')/(z\pi\mathcal{B}+z\pi'\mathcal{B}')$ can be seen to be very tractable. But we cannot claim a computational gain over just performing intersections. Symmetry reveals that:

$$z(\pi\mathcal{B} + \pi'\mathcal{B}')/(z\pi\mathcal{B} + z\pi'\mathcal{B}') = ((\pi\mathcal{B})_{mem} \cap (\pi'\mathcal{B}')_{mem})/(\pi\mathcal{B} \cap \pi'\mathcal{B}')_{mem}$$

And therefore:

Corollary 6.6.10.

$$((\iota\mathcal{B})_{mem} \cap (\iota'\mathcal{B}')_{mem})/\mathcal{B}^*_{mem} = ((\pi\mathcal{B})_{mem} \cap (\pi'\mathcal{B}')_{mem})/(\pi\mathcal{B} \cap \pi'\mathcal{B}')_{mem}$$

Proof. It follows immediately from Corollary 6.6.9 by manipulating quotients. □

However, the submodule $z\mathcal{B}$ in general contains less information than $\mathcal{B}$. Thus, there is some informational gain on that end.

6.6.4 Membership problems.

Let $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ be two systems, and let $s \in \mathcal{B}$ and $s' \in \mathcal{B}'$ be two signal such that $s[0] = s'[0] = a$. We then know that $a \in \mathcal{B}_{mem} \cap \mathcal{B}'_{mem}$. The question is: does a belong to $(\mathcal{B} \cap \mathcal{B}')_{mem}$? If $t = z^{-1}t' \in z^{-1}\mathbb{U}$, we will denote by zt the signal t'.

Proposition 6.6.11. *We have: $a \in (\mathcal{B} \cap \mathcal{B}')_{mem}$ if, and only if, $z(s - s') \in z\mathcal{B} + z\mathcal{B}'$.*

Proof. If $a \in (\mathcal{B} \cap \mathcal{B}')_{mem}$, then there is an $t \in \mathcal{B} \cap \mathcal{B}'$ with $t[0] = a$. In this case, we have $s - t \in \mathcal{B}$ and $t - s' \in \mathcal{B}'$. Then, as $t[0] = s[0]$, we get $z(s - t) \in z\mathcal{B}$. Similarly, we have $z(t - s') \in z\mathcal{B}'$. Then $z(s - s') = z(s - t) + z(t - s') \in z\mathcal{B} + z\mathcal{B}'$. Conversely, if $z(s - s') \in z\mathcal{B} + z\mathcal{B}'$ then $z(s - s') = w - w'$ for some $w \in z\mathcal{B}$ and $w' \in z\mathcal{B}'$. We then have $s - z^{-1}w = s' - z^{-1}w' = t$ with $t[0] = a$. □

Fully knowing a submodule of $\mathbb{U}$ requires at least having a basis for it. Using the proposition as a computational criterion assumes we can decide membership in modules. The theory of groebner bases may be used in these instances, but will not be considered in this chapter. Further insight ought to be derived from the characterization, by keeping track of suitable bases of the modules.

6.6.5 Properties on the role of memory.

We can extract more properties. The operator $\mathbb{H}$ captures the role of memory. If $\mathbb{H}(\mathbb{U}/\mathcal{B})$ is the 0 space, then memory in $(\mathbb{U}, \mathcal{B})$ does not usually have a role to play in the interaction.

Proposition 6.6.12. *If a system $(\mathbb{U}, \mathcal{B})$ is memoryless then $z\mathcal{B} = \mathcal{B}$.*

Proof. The statement follows by $(i.)$ in the definition of a memoryless system. □

The converse however is not true. Consider for example the LTI system $(\mathcal{R}, \langle 1 + z^{-1} + z^{-2} + \cdots \rangle)$ whose behavior consists of constant signals.

Corollary 6.6.13. *If a system $(\mathbb{U}, \mathcal{B})$ is memoryless then $\mathbb{H}(\mathbb{U}/\mathcal{B}) = 0$.*

Proof. We have $\mathbb{H}(\mathbb{U}/\mathcal{B}) = z\mathcal{B}/\mathcal{B}$. □

Some implications include:

Proposition 6.6.14. *If $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ are memoryless systems, then $(\mathcal{B} \cap \mathcal{B}')_{mem} = \mathcal{B}_{mem} \cap \mathcal{B}'_{mem}$, i.e., memory plays no role in the interaction.*

Proof. If $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ are memoryless, $(\mathbb{U}, \mathcal{B} + \mathcal{B}')$ is memoryless. Thus $z\mathcal{B} + z\mathcal{B}' = \mathcal{B} + \mathcal{B}' = z(\mathcal{B} + \mathcal{B}')$. □

Proposition 6.6.15. *If $\mathcal{B} + \mathcal{B}' = \mathbb{U}$, then $(\mathcal{B} \cap \mathcal{B}')_{mem} = \mathcal{B}_{mem} \cap \mathcal{B}'_{mem}$, i.e., memory plays no role in the interaction.*

Proof. If $\mathcal{B} + \mathcal{B}' = \mathbb{U}$, then $z\mathcal{B} + z\mathcal{B}' = \mathbb{U} = z(\mathcal{B} + \mathcal{B}')$. □

An LTI system $(\mathbb{U}, \mathcal{B})$ is said to be non-lagged if $z\mathcal{B} = \mathcal{B}$. In this case, we would have $\mathbb{H}(\mathbb{U}/\mathcal{B}) = 0$. A non-lagged system is one where no trajectory can be pulled back in time. Input-output systems provide a good source of non-lagged systems. Indeed, any system where $\mathcal{B} = \{(x, y) \in \mathcal{R}^{n+m} : y = Ax\}$ for some $\mathcal{R}$-matrix A can be seen as an input-output system, and can be shown to be non-lagged.

Proposition 6.6.16. *If $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ are non-lagged system, then the role of memory is* $\mathrm{lag}(\mathbb{U}, \mathcal{B} + \mathcal{B}')$.

Proof. The image of $\mathbb{H}(\mathbb{U}/\mathcal{B}) \oplus \mathbb{H}(\mathbb{U}/\mathcal{B}') \to \mathbb{H}(\mathbb{U}/(\mathcal{B} + \mathcal{B}'))$ is 0. □

Example 5. *Referring back to the running example:*

$$\textit{(A)} \quad x[n] = y[n] + y[n-1] \quad \textit{and} \quad \textit{(B)} \quad y[n] = x[n].$$

Both $(\mathbb{U}, \mathcal{B}_A)$ and $(\mathbb{U}, \mathcal{B}_B)$ are non-lagged, as they can be viewed as input-output devices. Thus, as stated in the introduction, one dimension in the memoryless system is lost due to memory because $lag(\mathcal{B}_A + \mathcal{B}_B)$ is one dimensional.

Proposition 6.6.17. *If $(\mathbb{U}, \mathcal{B} + \mathcal{B}')$ is non-lagged, then memory plays no role.*

Proof. The codomain of $\mathbb{H}(\mathbb{U}/\mathcal{B}) \oplus \mathbb{H}(\mathbb{U}/\mathcal{B}') \to \mathbb{H}(\mathbb{U}/(\mathcal{B} + \mathcal{B}'))$ is 0. □

One may go on to develop more similar insight, and generalize them to more complex system interconnection.

6.6.6 Destroying long-term memory.

As mentioned in the introduction, one may, naturally, forget only delays of length at least T, by keeping only information on signals up to time $T-1$. Specifically, we keep, from A and B of our running example, the pairs $\{(x[0, \cdots, T-1], y[0, \cdots, T-1])\}$ instead of $\{(x[0], y[0])\}$. The same phenomenon occurs, and the same techniques and solution apply.

Indeed, the module $\Phi(M)$ is obtained by tensoring M with $\mathcal{R}/z^{-1}\mathcal{R}$. We may then, instead, tensor M with $\mathcal{R}/z^{-T}\mathcal{R}$. This yields a system where only memory acting in less than T times steps remains in play. The procedure followed throughout the chapter, to characterize the role of memory, can be used unchanged for the new case. However, the modules obtained in the long exact sequence cannot be directly deduced from the exact sequence as done in Proposition 6.4.7. For instance, given an exact sequence of $\mathcal{R}$-module:

$$0 \to \mathcal{R}/z^{-1}\mathcal{R} \to M \to \mathcal{R}/z^{-1}\mathcal{R} \to 0,$$

the module M can be chosen to be either $\mathcal{R}/z^{-1}\mathcal{R} \oplus \mathcal{R}/z^{-1}\mathcal{R}$ or $\mathcal{R}/z^{-2}\mathcal{R}$. Of course, the module M cannot be any other choice than those two, but we cannot know which one it is with no additional information. An exact sequence does not necessarily *split* in the case of $\mathcal{R}$-modules as it does in the case of vector spaces. However, there is a means to deduce the *correct* module from the sequence. For instance, suppose we knew $\Phi(M)$. If $\Phi(M)$ has dimension 2, then $M = \mathcal{R}/z^{-1}\mathcal{R} \oplus \mathcal{R}/z^{-1}\mathcal{R}$ and if $\Phi(M)$ has dimension 1, then $M = \mathcal{R}/z^{-2}\mathcal{R}$. Such a means will not be pursued in this chapter.

6.7 Concluding remarks.

There exists a general principled method to recover the loss of exactness once we know that $\Phi(-)$ is tensoring $- \otimes \mathcal{R}/z^{-1}\mathcal{R}$. But the method is outside the scope of the chapter. We briefly touched upon it in Section 6.5.2, and we refer the reader to ([Wei95], Chap 2) for more details. We can however bypass explicitly following the method, as we have done. It reduces to our approach via the Snake lemma.

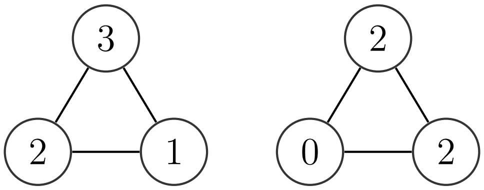
We did not work with basis of modules throughout the chapter. One needs to be careful as to whether we know the module as a submodule or just know it as a *general* module. As described in the preliminary section, two subobjects of an object (e.g., an $\mathcal{R}$-module) may be isomorphic as objects, but not as subobjects. Keeping track of bases and other computational requirements can be made effective through the use of Groebner bases.

The eventual goal is to apply the techniques of the chapter to settings of interest in cascading failure and contagion phenomena. The situation in the chapter generalizes almost immediately to settings where the objects in concern have a linear (or *abelian*) structure. In other setting, we would need to lift our objects of study to linear (or abelian) objects. We refer the reader to [Ada17a] for the details.

6.8 Appendix: Example on contagion.

Each system consists of a three node graph. Each node in the graph can be either black or white, and is assigned an integer k as a threshold. All nodes are white initially. A node then becomes black, if at least k of its neighbors are black. Once a node is black it remains black forever.

For instance, let A and B denote the systems on the left and right, respectively.

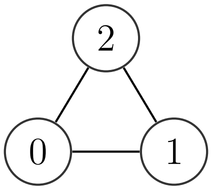


Given our rule above, a threshold of 0 indicates that a node *automatically* becomes black. If no threshold of 0 exists, then necessarily all nodes will remain white. We are interested in understanding the role that the evolution rules play when A and B interact.

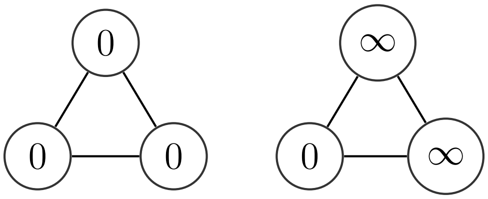
We can forget the evolution rules that are prone to interact with others by keeping from the systems only the set of final black nodes. Indeed, every set of black nodes

S corresponds to a decision-free system having a threshold of 0 on the nodes in S and a threshold of ∞ on the nodes not in S. Let us denote by A_{dec} and B_{dec} the decision-free systems derived from A and B respectively. Then the set of final black nodes of A_{dec} is empty, and that of B_{dec} contains the left node.

Two systems interact by combining their evolution rules. The system A&B corresponds to the graph that keeps on each node the minimum threshold between that of A and B:



Likewise, the system A_{dec}&B_{dec} corresponds to the graph that keeps on each node the minimum threshold between that of A_{dec} and B_{dec}. It can then be seen that $(\mathrm{A}\&\mathrm{B})_{dec}$ is different than A_{dec}&B_{dec}:



The evolution rules do play a role then, and we get an inequality similar to that presented in the case of linear systems.

When A and B are combined, the left black node in B interacts with the rules of A to color the right node black. Both the left and the right nodes then interact with the rules of B to color the middle node black. This effect is encoded in the inequality.

Chapter 7

Cascading phenomena in the behavioral approach

Abstract

This chapter studies the behavior of a subsystem as parts of its greater system undergo changes. As changes can lead, by means of interconnections, to changes in *remote* subsystems, the situation is inherently one that exhibits cascade-like effects. We cast the situation through the lens of the behavioral approach to systems theory, and recover a characterization relating the behavior of the subsystem to that of its greater system and the incurred change. We develop a short general theory to address the posed situation, and instantiate it to five cases: linear finite-dimensional systems, affine systems, finite systems, linear time-invariant systems and systems defined by polynomial equations. The theory relies on methods from homological algebra, and uncovers the *zero-dynamics* of a system as essential to relate the behavior of a subsystem to its greater system. The general pattern exhibited by the theory is of separate interest to understand interaction-related phenomena that generally occur in the interaction of systems.

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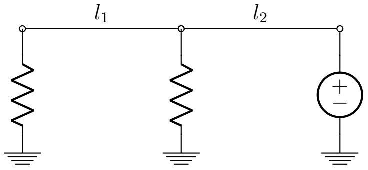
7.1 Introduction

Consider the following simple resistive circuit:



If line l_2 is disconnected from the source, then line l_1 will also be disconnected from the source. Indeed, the current passing through l_1 has to pass through l_2. The converse is, of course, not true. We have here a simple intuitive instance of a cascading phenomenon. How can we, however, make this intuition arise mathematically?

To formally set up the problem, we let l_i be in one of two states $\mathbb{U}_i = \{\text{on}_i, \text{off}_i\}$. The state *on* represents *connected to the source* and the state *off* represents *disconnected from the source*. The set of all states, termed the *universum*, is $\mathbb{U} := \mathbb{U}_1 \times \mathbb{U}_2$, and the set $\mathcal{B} \subseteq \mathbb{U}$ of admissible states, termed the *behavior*, is:

$$\mathcal{B} := \big\{(\text{on}_1, \text{on}_2), (\text{off}_1, \text{off}_2), (\text{off}_1, \text{on}_2)\big\}.$$

The set $\mathcal{B}$ indicates that if line l_2 is off_2, then line l_1 is necessarily off_1. One can incorporate time and delay into the situation, by considering states as timed trajectories. Regardless, nothing, up to this point in the formulation, is mathematically indicative of any cascading phenomenon.

7.1.1 The cascading intuition

We intuitively know that cascade-like effects occur because of the interrelation of the two lines. Such a statement is very informal. We can however mathematically test the intuition for cascading phenomena through the following experiment. Let us suppose that we are able to forget the interrelation between l_1 and l_2 in $\mathcal{B}$ to get $\mathcal{B}_{unrelated}$. We can then compare the separate responses of $\mathcal{B}$ and $\mathcal{B}_{unrelated}$ towards a change in the state of l_2. Cascade-like phenomena would be in play if there is a discrepancy between those responses. If we let l_2 become off in $\mathcal{B}$, then l_1 can only be off. However, as we shall observe, if we let l_2 become off in $\mathcal{B}_{unrelated}$, then l_1 can be either on or off.

If we formalize going from $\mathcal{B}$ to $\mathcal{B}_{unrelated}$ as a certain function, the phenomenon becomes encoded in the function. Specifically, let π_1 and π_2 denote the canonical projections onto the coordinates:

$$\pi_1 : \mathbb{U} \to \mathbb{U}_1 \text{ and } \pi_2 : \mathbb{U} \to \mathbb{U}_2.$$

We can forget the interrelation between the lines l_1 and l_2 by defining a map:

$$\pi : \mathcal{B} \longmapsto \pi_1\mathcal{B} \times \pi_2\mathcal{B}.$$

As $\pi_1\mathcal{B} = \mathbb{U}_1$ and $\pi_2\mathcal{B} = \mathbb{U}_2$, we get $\pi\mathcal{B} = \mathbb{U}$. The system $\pi\mathcal{B}$ is $\mathcal{B}_{unrelated}$, and ought to be interpreted as the *best* system describing $\mathcal{B}$ if the lines were forced to be independent. Let us then suppose that we disconnect line l_2. This is achieved by declaring only $(*, \text{off}_2)$ as allowable. Equivalently, this is achieved mathematically by intersecting $\mathcal{B}$ with $\mathcal{C} := \{(\text{on}_1, \text{off}_2), (\text{off}_1, \text{off}_2)\}$. When performing the change, we get:

$$\pi(\mathcal{B} \cap \mathcal{C}) \neq \pi(\mathcal{B}) \cap \pi(\mathcal{C}).$$

The presence of the inequality is indicative of the phenomenon. The phenomenon arises if, and only if, the map π fails to commute with $\cap$. **Our goal is to understand and characterize the inequality**. Understanding the inequality, can be done by separately understanding the map π_1 and π_2. Our interest then reduces to understanding the effect of change on a subsystem of the greater system.

7.1.2 The general one-sided situation

We then consider a mega-system comprised of an interacting mixture of infrastructures (e.g., power, transportation, communication), markets (e.g., prices, firms, consumers), political entities and many individuals. We are interested in understanding the evolution of the behavior of a subsystem of this mega-system, as changes are effected into the mega-system. Of courses, changes directly effected onto the subsystem modifies its behavior. It is also the case that seemingly non-related changes causes a shift in the behavior by a successive chain of events.

Let $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{R}$ be sets such that $\mathbb{M} = \mathbb{S} \times \mathbb{R}$. Following the behavioral approach to systems theory terminology (see e.g., [Wil07], [PW98], also later on described in Subsection 7.2.1) we will have $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{R}$ be the outcome space, or universum, of the mega-system, the subsystem, and the rest (or remainder) in the mega-system that is not the subsystem of interest. The systems will then be subsets of those universa. Specifically, the sets $\mathcal{M} \subseteq \mathbb{M}$, $\mathcal{S} \subseteq \mathbb{S}$ and $\mathcal{R} \subseteq \mathbb{R}$ denote the behavior of the mega-system, the subsystem and the rest, respectively. Although $\mathcal{S}$ is a subsystem of $\mathcal{M}$, the set $\mathcal{S}$ is not a subset of $\mathcal{M}$, but is rather a projection (or a quotient) of $\mathcal{M}$ onto the $\mathbb{S}$-coordinate. A change in our mega-system, following the behavioral approach, is then depicted as an intersection with a behavior $\mathcal{C} \subseteq \mathbb{M}$. If we denote by $\pi : \mathbb{M} \to \mathbb{S}$ the projection onto the $\mathbb{S}$-coordinate, then $\pi\mathcal{M} = \mathcal{S}$ and we generally observe:

$$\pi(\mathcal{M} \cap \mathcal{C}) \neq \pi(\mathcal{M}) \cap \pi(\mathcal{C}).$$

The change $\mathcal{C}$ affects the subsystem $\mathcal{S}$ through interactions with $\mathcal{R}$. If all the interactions were confined to be within $\mathcal{R}$, then equality would follow. In such a case, changes outside of $\mathcal{S}$ do not affect $\mathcal{S}$.

7.1.3 The question and the contribution

Thus, as already motivated by the example, the inequality is synonymous to the presence of cascade-like phenomena. The question we then ask is:

Question. *How can we non-trivially relate $\pi(\mathcal{M} \cap \mathcal{C})$ to $\pi(\mathcal{M})$, $\pi(\mathcal{C})$ and potentially properties, or features, of $\mathcal{M}$ and $\mathcal{C}$ despite the presence of the inequality?*

Informally, how can we relate the behavior of the subsystem, to that of its greater system and the incurred change? We develop a short general theory to answer that question, and instantiate it to the following main cases:

- Linear (finite-dimensional) systems over fields.
- Affine systems over fields.
- Finite systems.
- Linear time-invariant systems.
- Algebraic systems defined by polynomial equations.

In each case, we obtain a different characterization following a same general pattern. In linear systems, where $\mathcal{M}$ and $\mathcal{C}$ are subspaces of a vector space $\mathbb{M}$, we retrieve, for instance, a characterization of the form:

$$(\pi\mathcal{M} \cap \pi\mathcal{C})/\pi(\mathcal{M} \cap \mathcal{C}) = \rho(\mathcal{M} + \mathcal{C})/(\rho\mathcal{M} + \rho\mathcal{C}).$$

The space $\pi\mathcal{M}$ is a subspace of $\mathbb{S}$, whereas $\rho(\mathcal{M})$ is a subspace of $\mathbb{R}$. The quantity $\rho(\mathcal{M})$ is an invariant of $\mathcal{M}$, smaller than $\mathcal{M}$, that captures the potential for cascading behavior. The space $\rho(\mathcal{M})$ corresponds, in this case, to the subspace of $\mathcal{M}$ having 0 as an $\mathbb{S}$-coordinate. If $\pi(\mathcal{M})$ is interpreted (or labeled) as the output of the system $\mathcal{M}$, then $\rho(\mathcal{M})$ corresponds to the *zero-dynamics* of the system (see, e.g., [Isi95] Ch 4, Sec 4.3 and [Isi13]) interpreted through the lens of the behavioral approach. The link to zero-dynamics and its potential implications are not pursued in this paper, but may be a subject of further investigation. The link is however briefly revisits in Section 7.4.2.

The characterization can be refined, and lends itself to a variety of consequences. It is generally attained through the following observation:

i. Interconnection (or change) in systems can be expressed as an exact sequence.

ii. Cascading phenomena is synonymous to a loss of exactness.

iii. Exactness can be recovered by using adequate invariants.

The characterization relies on methods in homological algebra.

In mathematical terms, the paper amounts to developing a (co)homology theory capture the effects of the cascade-like phenomenon. The 0th order (co)homology object encodes the behavior of the subsystem, and the higher (co)homology objects (in this paper only one is non-trivial) encode the potential of a system for effects. Such notions will not be explicitly used in this paper, but may be helpful to keep in mind, if one is familiar with the terms.

7.1.4 Outline

We begin, in Section 2, with a review of the behavioral approach and a development of the situation of cascade-like effects at hand. We then develop and present the general theory in Section 3. We instantiate it to linear (finite-dimensional) systems over fields, affine systems over fields, finite systems, linear time-invariant systems and algebraic systems defined by polynomial equations in sections 4, 5, 6, 7 and 8, respectively. These sections will show an intentional recurring pattern. We finally conclude with some remarks in Section 9.

7.2 Preliminaries

We assume the reader is familiar with basic elements of commutative algebra, namely commutative rings and modules. They will be used on an elementary level. We refer the reader to [AM69] for details on the matter.

7.2.1 Review of the behavioral approach

Rather than viewing a system as an input-output device, the behavioral approach views a system as a collection of trajectories allowed possible by the laws of a model.

Definition 7.2.1 (cf. [PW98], Section 1.2.1)**.** *A Willems system is a pair* $(\mathbb{U}, \mathcal{B})$ *where* $\mathbb{U}$ *is a set, called the universum—its elements are called* outcomes*—and* $\mathcal{B}$ *a subset of* $\mathbb{U}$ *called the* behavior*.*

A system is made dynamical by considering universa of the form $\mathbb{W}^{\mathbb{T}}$, the set of maps from a set $\mathbb{T}$ to a set $\mathbb{W}$. Linearity and time-invariance emerge when the universa and the behaviors are endowed with a certain structure. The behavioral approach also enables us to define interconnections of systems. Two systems are interconnected to yield a behavior containing the outcomes that are allowed possible by both systems. Indeed, if the behaviors are solutions to sets of equations, then the interconnection informally consists of merging, or combining, the sets of equations together. We then have:

Definition 7.2.2. *The interconnection of* $(\mathbb{U}, \mathcal{B})$ *and* $(\mathbb{U}, \mathcal{B}')$ *yields the system* $(\mathbb{U}, \mathcal{B} \cap \mathcal{B}')$*.*

We refer the reader to [PW98] and [Wil07] for more details on the physical interpretation of the approach.

Remark All the cases studied in this paper are attained by equipping the universum $\mathbb{U}$ and the behaviors $\mathcal{B}$ with suitable mathematical structures. When setting up the general problem, we can however forget about the structure and consider only the sets underlying their systems. Most importantly, forgetting the structure will not cause any problems when it comes to interconnections. For instance, the intersection of two subspaces of a vector space amounts to intersecting their underlying sets. One reason for this is because the algebraic structures considered are the fixed-points of closure operators on the set of subsets. Those fixed-points are always closed under set-intersection.

7.2.2 The cascading intuition for a subsystem

Recall that $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{R}$ are sets, defining the universa of the mega-system, the subsystem and the rest, respectively. As $\mathbb{M} = \mathbb{S} \times \mathbb{R}$, we have a canonical projection, namely a surjective map:

$$\pi : \mathbb{M} \longrightarrow \mathbb{S}.$$

Let $2^{\mathbb{M}}$ denote the power set of $\mathbb{M}$. Every element M of $2^{\mathbb{M}}$ then defines a (mega)system $(\mathbb{M}, M)$ in the Willems sense. Similarly, every $S \in 2^{\mathbb{S}}$ defines a (sub)system in the Willems sense. The megasystems in $2^{\mathbb{M}}$ are related via an inclusion order relation $\subseteq$, and two megasystems M and M' are interconnected, in a unique way, to yield their greatest lower-bound $M \cap M'$. The same holds for $2^{\mathbb{S}}$.

If $(2^{\mathbb{M}}, \cap)$ and $(2^{\mathbb{S}}, \cap)$ denote the semilattices of megasystems and subsystems, respectively, then the projection π lifts to a map:

$$\pi : (2^{\mathbb{M}}, \cap) \longrightarrow (2^{\mathbb{S}}, \cap)$$

where explicitly:

$$\pi : M \longmapsto \pi(M) := \{s \in \mathbb{S} : (s, r) \in \mathbb{M} \text{ for some } r \in \mathbb{R}\}$$

The map π satisfies two properties:

P.1. If $M \subseteq M'$, then $\pi(M) \subseteq \pi(M')$.

P.2. For every $S \in 2^{\mathbb{S}}$, there exists a largest $M \in 2^{\mathbb{M}}$ such that $\pi(M) = S$.

The first property states that π is order-preserving, thus preserving the relationship among megasystems when going to subsystems. The second property states that every possible subsystem S can be completed into a mega-system in a simplest possible way, namely into $S \times \mathbb{R}$. We may think of the map π as *a veil* that conceals parts or mechanisms in the mega-system and leaves the subsystem observable. Cascading phenomena, or more precisely what we term *generative effects* (in [Ada17a] and [Ada17b]), are said to emerge when the result of the interaction cannot be explained by the observable part only. Specifically, when:

$$\pi(M \cap M') \neq \pi(M) \cap \pi(M'),$$

for some M and M'. The intuition of cascade-like phenomena is manifested in the inequality. The goal of this paper is to non-trivially relate $\pi(M \cap M')$ to $\pi(M) \cap \pi(M')$. We refer the reader to [Ada17a] and [Ada17b] for more details on generative effects.

7.2.3 Cascading intuition among multiple subsystems

Let us suppose that $\mathbb{M} = \mathbb{S}_1 \times \cdots \times \mathbb{S}_n$. Every factor may be seen to correspond to a component of our mega-system. We may attempt to capture the cascade-like effects that may emerge from the interaction of these systems, through the following approach, already described in the introduction.

For each component i, we define $\pi_i : \mathbb{M} \to \mathbb{S}_i$ to be the canonical projection. As seen in the previous subsection, π_i is a veil and will generally result in an inequality of the form:

$$\pi_i(M \cap M') \neq \pi_i(M) \cap \pi_i(M'),$$

for some M and M'. The inequality indicates that what is concealed (by π_i) in the system, can affect, when changed, the behavior of the ith component. We can then define π to be product of the π_i's. Namely:

$$\pi : M \longmapsto \pi_1(M) \times \cdots \times \pi_n(M).$$

The map π destroys all the interrelation among the components, and yields the simplest system if all the components were not allowed to interact. As a consequence of the separate π_i's, we generally get:

$$\pi(M \cap M') \neq \pi(M) \cap \pi(M'),$$

The analysis, in this paper, is intended to aid in understanding this situation better. More specifically, it aims to quantify the inequality. However, as already mentioned, such a study can be performed on the separate π_i's. The paper will then only study the one-sided situation.

7.3 The general theory

Throughout this section, $\mathcal{R}$ will be a commutative ring with unit. The notion of exact sequences will be central to the paper. Interconnection of systems will be expressed as an exact sequence. The presence of cascading phenomena will be linked to a loss of exactness. Finally, relating the subsystem of the megasystem to the separate components will be realized by an exact sequence.

7.3.1 Exact sequences of $\mathcal{R}$-modules

We begin by the notion of exactness in sequences of $\mathcal{R}$-modules.

Definition 7.3.1. *A sequence of $\mathcal{R}$-modules M_i and $\mathcal{R}$-module homomorphisms f_i*

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

is said to be exact at M_i if $\operatorname{im} f_i = \ker f_{i+1}$. The sequence is called an exact sequence if it is exact at every M_i.

The kernel of a linear map $f : M \to N$ is the largest submodule K of M such that $f(K) = 0$. Two items are then in play to define the kernel: the object K and the inclusion relation as a submodule. It is thus beneficial to explicitly think of the kernel as an injective map $K \to M$. As a clarification:

Definition 7.3.2. *Let $f : M \to N$ be a linear map. The kernel, image and cokernel of f are respectively the canonical linear maps $\ker(f) \to M$, $\operatorname{im}(f) \to N$ and $N \to \operatorname{coker}(f)$. We define $\ker(f)$, $\operatorname{im}(f)$ and $\operatorname{coker}(f)$ to be the objects of the kernel, image and cokernel of f.*

In light of this:

Proposition 7.3.3. *The sequence $0 \to \operatorname{im}(f) \to N \to \operatorname{coker}(f) \to 0$ is an exact sequence for every $\mathcal{R}$-linear map $f : M \to N$.*

Proof. We have that $\operatorname{im}(f) \to N$ is injective and that $N \to \operatorname{coker}(f)$ is surjective. As $\operatorname{coker}(f) = N/\operatorname{im}(f)$, we get exactness at N. □

Or dually:

Proposition 7.3.4. *The sequence $0 \to \ker(f) \to M \to \operatorname{im}(f) \to 0$ is an exact sequence for every $\mathcal{R}$-linear map $f : M \to N$.*

Proof. We have $\ker(f) \to M$ is injective and $M \to \operatorname{im}(f)$ is surjective. Exactness at M follows from the first isomorphism theorem: $\operatorname{im}(f) = M/\ker(f)$. □

The image $\operatorname{im}(f)$ by duality should be the coimage $\operatorname{coim}(f)$. However, the the two objects are isomorphic in the case of $\mathcal{R}$-modules, or other *abelian* objects in general.

Interconnection of systems will occur from an instance of Proposition 7.3.4. To illustrate, let us consider each system to be an $\mathcal{R}$-module, rather than a pair $(\mathbb{U}, \mathcal{B})$. We are then given two maps $f : M \to N$ and $f' : M' \to N$. The modules M and M'

correspond to our systems, and the maps f and f' correspond to how the systems will be *glued* along the system N. The interconnected system then corresponds to $\ker(f - f')$ and we recover an exact sequence:

$$0 \longrightarrow \ker(f - f') \longrightarrow M \oplus M' \longrightarrow \operatorname{im}(f - f') \longrightarrow 0$$

In particular, following the behavioral approach, let $\mathbb{U}$ be a module. Suppose M and M' are submodules of $\mathbb{U}$. We then have two Willems systems $(\mathbb{U}, M)$ and $(\mathbb{U}, M')$, and two canonical injections:

$$f : M \to \mathbb{U} \text{ and } f' : M' \to \mathbb{U}$$

The module $M \oplus M'$ does not yield a submodule of $\mathbb{U}$. *Gluing* them along the injections f and f', however, yields a submodule of $\mathbb{U}$. We then recover an exact sequence:

$$0 \longrightarrow \ker(f - f') \longrightarrow M \oplus M' \xrightarrow{f-f'} \operatorname{im}(f - f') \longrightarrow 0$$

The modules $\operatorname{im}(f - f')$ and $\ker(f - f')$ are $M + M'$ and $M \cap M'$, respectively. We just recovered the canonical exact sequence $0 \to M \cap M' \to M \oplus M' \to M + M' \to 0$.

7.3.2 Exact sequences of linear maps

The notion of exactness also extends to sequences of linear maps. We first need to define a notion of linear map homomorphism. Let $f : M \to N$ and $f' : M' \to N'$ be $\mathcal{R}$-linear maps.

Definition 7.3.5. *A morphism $\Phi : f \Rightarrow f'$ is pair of maps $\phi_M : M \to M'$ and $\phi_N : N \to N'$ such that the diagram:*

$$\begin{tikzcd}
M \arrow[r, "\phi_M"] \arrow[d, "f"] & M' \arrow[d, "f'"] \\
N \arrow[r, "\phi_N"] & N'
\end{tikzcd}$$

commutes.

Given a morphism $\Phi : f \Rightarrow f'$, we may then define kernels, images and cokernels. There exists, however, two ways to define such notions for a square diagram: either *vertically* or *horizontally.* The horizontal notion will be used to extend the notion of exactness to sequences of linear maps. The vertical notion will then be used to send exact sequences of linear maps to sequences (non-necessarily exact) of $\mathcal{R}$-modules.

Definition 7.3.6. *Let $\Phi := (\phi_M, \phi_N)$ be a morphism. The objects of the **horizontal** kernel, image and cokernel of Φ are, respectively, the canonical linear maps $\ker_h \Phi : \ker \phi_M \to \ker \phi_N$, $\operatorname{im}_h \Phi : \operatorname{im} \phi_M \to \operatorname{im} \phi_N$ and $\operatorname{coker}_h \Phi : \operatorname{coker} \phi_M \to \operatorname{coker} \phi_N$.*

Indeed, recall that the (horizontal) kernel (resp. image, cokernel) of $\Phi : f \Rightarrow f'$ is a linear map $\ker_h \Phi \Rightarrow f$ (resp. $\mathrm{im}_h \Phi \Rightarrow f$, $f' \Rightarrow \mathrm{coker}_h \Phi$). We may then extend the notion of exact sequences to linear maps and their morphisms:

Definition 7.3.7. *A sequence of $\mathcal{R}$-linear maps f_i and morphisms Φ_i*

$$\cdots \Longrightarrow f_{i-1} \xRightarrow{\Phi_i} f_i \xRightarrow{\Phi_{i+1}} f_{i+1} \Longrightarrow \cdots$$

is said to be exact at f_i if $\mathrm{im}_h \Phi_i = \ker_h \Phi_{i+1}$. The sequence is called an exact sequence if it is exact at every f_i.

Proposition 7.3.8. *A sequence of $\mathcal{R}$-linear maps $f_i : M_i \to N_i$ and morphisms Φ_i*

$$\cdots \Longrightarrow f_{i-1} \xRightarrow{\Phi_i} f_i \xRightarrow{\Phi_{i+1}} f_{i+1} \Longrightarrow \cdots$$

is exact at M_i if, and only if, the commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & M_{i-1} \arrow[r, "\phi_{M,i}"] \arrow[d, "f_{i-1}"'] & M_i \arrow[r, "\phi_{M,i+1}"] \arrow[d, "f_i"'] & M_{i+1} \arrow[r] \arrow[d, "f_{i+1}"'] & \cdots \\
\cdots \arrow[r] & N_{i-1} \arrow[r, "\phi_{N,i}"] & N_i \arrow[r, "\phi_{N,i+1}"] & N_{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

has rows exact at M_i and N_i. The sequence is exact if the rows of the diagram are exact sequences.

Proof. The condition $\mathrm{im}_h \Phi_i = \ker_h \Phi_{i+1}$ is to be checked pointwise at M_i and N_i. □

An exact sequence of $\mathcal{R}$-linear maps is only a commutative ladder with exact rows. Similarly:

Definition 7.3.9. *Let $\Phi : f \Rightarrow f'$ be a morphism. The objects of the* ***vertical*** *kernel, image and cokernel of Φ are, respectively, the canonical linear maps $\ker_v \Phi : \ker f \to \ker f'$, $\mathrm{im}_v \Phi : \mathrm{im}\, f \to \mathrm{im}\, f'$ and $\mathrm{coker}_v \Phi : \mathrm{coker}\, f \to \mathrm{coker}\, f'$.*

The vertical kernel, image and cokernel will be used to operate on sequences of linear maps.

7.3.3 The loss of exactness

Every $\mathcal{R}$-linear map f admits a cokernel, whose object is $\mathrm{coker}(f)$. The coker operator also acts on morphsims $\Phi : f \Rightarrow f'$ of $\mathcal{R}$-linear maps, sending Φ to a linear map $\mathrm{coker}_v(\Phi)$ as discussed in the previous subsection, in such a way that:

i. For all $\Phi : f \Rightarrow f'$ and $\Phi' : f' \Rightarrow f''$, we have $\mathrm{coker}_v(\Phi \circ \Phi') = \mathrm{coker}_v(\Phi) \circ \mathrm{coker}_v(\Phi')$.

ii. If $\mathrm{id}_f : f \Rightarrow f$ is the identity morphism, then $\mathrm{coker}_v(\mathrm{id}_f) = \mathrm{id}_{\mathrm{coker}(f)}$ is the identity linear map on $\mathrm{coker}(f)$.

An operator satisfying the two properties is termed a **functor**. Not only does it acts on objects, but also on morphisms between objects.

Remark 7.3.10. *We hereafter drop the v subscript, and denote* coker_v *(resp.* $\ker_v$, im_v*) by* coker *(resp.* ker, im*). We however always refer to the horizontal notions* coker_h, $\ker_h$ *and* im_h *using the corresponding h subscript.*

The functor coker is also said to be *additive* as:

Proposition 7.3.11. *If f and f' are linear maps, then* $\mathrm{coker}(f \oplus f') = \mathrm{coker}(f) \oplus \mathrm{coker}(f')$.

Proof. Let M and M' be the domain of f and f' respectively. The sequence $0 \to \mathrm{im}(f) \oplus \mathrm{im}(f') \to M \oplus M' \to \mathrm{coker}(f) \oplus \mathrm{coker}(f) \to 0$ is exact. As $\mathrm{im}(f \oplus f) = \mathrm{im}(f) \oplus \mathrm{im}(f')$ and cokernels are unique up to isomorphism, we get $\mathrm{coker}(f \oplus f') = \mathrm{coker}(f) \oplus \mathrm{coker}(f')$. □

As coker acts on both linear maps and their morphisms, it can be applied to an exact sequence:

Proposition 7.3.12. *If $0 \Rightarrow f \Rightarrow f' \Rightarrow f'' \Rightarrow 0$ is a short exact sequence of linear maps, then the sequence* $0 \to \mathrm{coker}\, f \to \mathrm{coker}\, f' \to \mathrm{coker}\, f'' \to 0$ *is always exact at* $\mathrm{coker}\, f''$ *and* $\mathrm{coker}\, f'$.

Proof. See e.g., Proposition 7.3.14. □

The coker functor does not however preserve exactness on the left. Indeed, the map $\mathrm{coker}\, f \to \mathrm{coker}\, f'$ may fail to be injective.

Proposition 7.3.13. *If $0 \Rightarrow f \Rightarrow f' \Rightarrow f'' \Rightarrow 0$ is a short exact sequence of linear maps, then the sequence* $0 \to \mathrm{coker}\, f \to \mathrm{coker}\, f' \to \mathrm{coker}\, f'' \to 0$ *need not be exact at* $\mathrm{coker}\, f$.

Proof. See e.g., Proposition 7.3.14. □

This loss of exactness is the cause of the cascading phenomena as we shall see.

7.3.4 The ker-coker sequence

The ker operator is also a functor, acting on both $\mathcal{R}$-linear maps and their morphisms. The important feature of ker is that it complements the information of the coker functor in the following sense:

Proposition 7.3.14 (Corollary of the snake lemma, cf. [AM69] ch. 2, p. 23)**.** *Let $0 \Rightarrow f \Rightarrow f' \Rightarrow f'' \Rightarrow 0$ be a short exact sequence, there exists a map δ such that the sequence:*

$$0 \longrightarrow \ker f \longrightarrow \ker f' \longrightarrow \ker f'' \xrightarrow{\delta} \mathrm{coker}\, f \longrightarrow \mathrm{coker}\, f' \longrightarrow \mathrm{coker}\, f'' \longrightarrow 0$$

is exact.

Proof. The statement follows from a direct application of the snake lemma, see e.g. [AM69] ch. 2, p. 23, proposition 2.10. □

The pair (coker, ker) is termed a δ-functor (see e.g., [Wei95] ch 2. for the terminology). The kernel functor encodes what is causing the loss of exactness of the cokernel functor.

Given such a six-term exact sequence, we may use it to relate elements of it to its other constituents. Specifically, in the case of vector spaces, we get:

Proposition 7.3.15. *If the sequence of vector spaces:*

$$0 \longrightarrow V_0 \longrightarrow V_1 \xrightarrow{f} V_2 \longrightarrow V_3 \longrightarrow V_4 \xrightarrow{g} V_5 \longrightarrow 0,$$

is exact, then $V_0 = \ker f$ *and* $V_3 = \operatorname{coker} f \oplus \ker g$*.*

Proof. The sequence $0 \to \operatorname{im}(V_1 \to V_2) \to V_2 \to \operatorname{im}(V_2 \to V_3) \to 0$ is exact. Indeed, we have $\operatorname{im}(V_1 \to V_2) = V_1/\ker(V_1 \to V_2)$. As $\ker(V_1 \to V_2) = \operatorname{im} f$ by exactness of the six-term sequence, we get that $\operatorname{im}(V_1 \to V_2) = \operatorname{coker}(f)$. By exactness, we also get $\operatorname{im}(V_2 \to V_3) = \ker g$. Finally, short exact sequence of vectors spaces split. Namely, if $0 \to U \to V \to W \to 0$ is a sequence of vector spaces, then $V = U \oplus W$. □

If the elements of the sequence are not vector spaces, and we are given no additional information, then we can only deduce short exact sequences:

Proposition 7.3.16. *If the sequence of* $\mathcal{R}$*-modules* $M_1 \xrightarrow{f} M_2 \to M_3 \to M_4 \xrightarrow{g} M_5$ *is exact, then the sequence* $0 \to \operatorname{coker} f \to M_3 \to \ker g \to 0$ *is exact.*

Proof. We have $\ker g = \operatorname{im}(M_3 \to M_4)$ by exactness. We also have $\operatorname{coker} f = M_2/\operatorname{im}(f)$ and $\operatorname{im} f = \ker(M_2 \to M_3)$. By the first isomorphism theorem, we get $\operatorname{im}(M_2 \to M_3) = M_2/\ker(M_2 \to M_3)$. □

With additional information on the sequence, or on the situation it arises from, we may deduce M_3 accordingly.

As will become clear as the paper goes along, the ker functor throws away information from linear maps, and keeps only what is necessary to understand how much exactness is lost. Conversely, we can interpret ker as throwing everything that is not needed to recover the exactness lost, via the following:

Proposition 7.3.17. *Let* (coker, G) *be a* δ*-functor (see e.g., [Wei95] Ch 2 for a complete definition). There exists a unique collection of* $\mathcal{R}$*-linear maps* $\alpha_f : G(f) \to \ker(f)$*, one for each* f*, such that, for every exact sequence* $0 \Rightarrow f \Rightarrow f' \Rightarrow f'' \Rightarrow 0$*, the diagram:*

$$\begin{tikzcd}
0 \arrow[r] & G(f) \arrow[r] \arrow[d, "\alpha_f"'] & G(f') \arrow[r] \arrow[d, "\alpha_{f'}"'] & G(f'') \arrow[r] \arrow[d, "\alpha_{f''}"'] & \operatorname{coker}(f) \arrow[r] \arrow[d, "\simeq"'] & \operatorname{coker}(f') \arrow[r] \arrow[d, "\simeq"'] & \operatorname{coker}(f'') \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & \ker(f) \arrow[r] & \ker(f') \arrow[r] & \ker(f'') \arrow[r] & \operatorname{coker}(f) \arrow[r] & \operatorname{coker}(f) \arrow[r] & \operatorname{coker}(f) \arrow[r] & 0
\end{tikzcd}$$

commutes.

Proof. The proof is outside the scope of the paper, we refer to [Ada17g] for more details. The statement amounts to showing that $(\mathrm{coker}, \ker)$ is a universal δ-functor (see e.g., [Wei95] ch. 2 for the terminology). □

Most importantly:

Proposition 7.3.18. *For every $\mathcal{R}$-linear map f, the map $\alpha_f : G(f) \to \ker(f)$ is surjective.*

Proof. The proof is outside the scope of the paper, we refer to [Ada17g] for a proof and more details. □

The opposite theory By duality, the last two sections can be developed by interchanging the words kernel and cokernel. All surjective maps would become injective maps, and vice versa. The ker functor will not preserve exactness on the right. The coker encodes the additional information needed to recover exactness. The pair $(\ker, \mathrm{coker})$ is also a δ-functor. The pair $(\ker, \mathrm{coker})$ is specifically known as a cohomological δ-functor, whereas the pair $(\mathrm{coker}, \ker)$ is known as a homological δ-functor.

7.3.5 How to apply it to systems?

The key result to be used is the six-term exact sequence relating the kernels of linear maps to their cokernels. To apply the theory to our classes of systems, we need to lift *our situation* to a linear map. Such a lift consists of transforming our situation of understanding the subsystem of a system into a situation of recovering the cokernel (or kernel) of a linear map, as proposed by the above theory. A lift can then be seen as a functor, that sends a system to a linear map and an inclusion of systems into a morphism of linear maps. The lift will be successful if it adheres to the following three properties. First, it encodes the behavior of the subsystem as either the kernel or the cokernel. Second, it preserves the interconnection of systems. Third, it encodes the cascade-like effects in a loss of exactness. If the lift is successful, then applying the solution of the above theory will give us a meaningful result in our situation.

7.4 Linear systems over fields

Let k be a field. We begin by considering linear systems defined over the field k.

Definition 7.4.1. *A Willems system $(\mathbb{U}, \mathcal{B})$ is said to be k-linear if $\mathbb{U}$ is a k-vector space, and $\mathcal{B}$ is a linear subspace of $\mathbb{U}$. The system $(\mathbb{U}, \mathcal{B})$ is additionally said to be finite-dimensional if $\mathbb{U}$ has finite dimension.*

Our systems $(\mathbb{M}, M)$, $(\mathbb{S}, S)$ and $(\mathbb{R}, R)$ will be finite-dimensional k-linear systems. Let $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{R}$ be finite-dimensional k-vector space. The veil $\pi : \mathbb{M} \to \mathbb{S}$ is a surjective linear map that projects an element $m = (s, r) \in \mathbb{M}$ into an element $s \in \mathbb{S}$.

7.4.1 A lift to linear maps

We will lift our systems to linear maps as follows:

Definition 7.4.2. *We define L to be a functor that sends every subspace M of $\mathbb{M}$ to:*

$$\begin{tikzcd} h_M := & \mathbb{M} \arrow[d, "{(\pi,p)}"] \\ & \mathbb{S} \oplus (\mathbb{M}/M) \end{tikzcd}$$

and every order relation $M \subseteq M'$, or equivalently inclusion map $M \to M'$ to a morphism $h_M \Rightarrow h_{M'}$ of linear maps:

$$\begin{tikzcd} \mathbb{M} \arrow[r, "id"] \arrow[d, "{(\pi,p)}"'] & \mathbb{M} \arrow[d, "{(\pi,p')}"] \\ \mathbb{S} \oplus (\mathbb{M}/M) \arrow[r, "{(\mathrm{id},i)}"'] & \mathbb{S} \oplus (\mathbb{M}/M') \end{tikzcd}$$

The lift encodes the behavior of the subsystem in the cokernel of the map:

Proposition 7.4.3. *For every M, we have* $\operatorname{coker} h_M = \mathbb{S}/\pi(M)$.

Proof. The image of (π, p) is $\pi(M) \oplus \mathbb{M}/M$. □

The lift also preserves the construction, or interconnection, of systems:

Proposition 7.4.4. *The sequence $0 \Rightarrow h_{M\cap M'} \Rightarrow h_M \oplus h_{M'} \Rightarrow h_{M+M'} \Rightarrow 0$ is exact for every M and M'.*

Proof. The sequence $0 \to \mathbb{M}/(M \cap M') \to \mathbb{M}/M \oplus \mathbb{M}/M' \to \mathbb{M}/(M + M') \to 0$ is exact. Direct sum of exact sequences yields an exact sequence. The two rows of the obtained ladder diagram are then easily seen to be exact. □

Finally, cascade-like phenomena occur precisely when there is loss of exactness:

Proposition 7.4.5. *For every M and M', we have $\pi(M\cap M') \neq \pi(M)\cap\pi(M')$ if, and only if, the sequence $0 \to \operatorname{coker} h_{M\cap M'} \to \operatorname{coker} h_M \oplus \operatorname{coker} h_{M'} \to \operatorname{coker} h_{M+M'} \to 0$ is not exact at $\operatorname{coker} h_{M\cap M'}$.*

Proof. We have $\operatorname{coker} h_{M\cap M'} = \mathbb{S}/\pi(M \cap M')$. Exactness on the left is then lost if, and only if, $\pi(M \cap M') \neq \pi(M) \cap \pi(M')$. □

The object encoding the potential for this loss is the kernel.

Proposition 7.4.6. *We have $\ker h_M = \{r \in \mathbb{R} : (0, r) \in M\}$ is a subspace of $\mathbb{R}$.*

Proof. We have $x \in \ker h_M$ if, and only if, $\pi(x) = 0$ and $x \in M$. □

The information we need consists of forcing all the values in the subsystem to be zero, and backtracking to the potential values in the remaining part of the megasystem.

7.4.2 The potential for cascades

Let us define ρ (for *reduce*) to be the map sending a subspace M into $\rho(M) = \ker h_M$. Namely:

Definition 7.4.7. *We define ρ such that $M \mapsto \{r \in \mathbb{R} : (0, r) \in M\}$.*

The map ρ can be seen to be a dual of π in the following sense:

I.1. If $M \subseteq M'$, then $\rho(M) \subseteq \rho(M')$.

I.2. For every $R \in 2^{\mathbb{R}}$, there exists a smallest $M \in 2^{\mathbb{M}}$ such that $\rho(M) = R$.

Another duality comes from the snake lemma, as we get the following theorem:

Theorem 7.4.8. *For every M and M', we have:*

$$\mathbb{S}/\pi(M \cap M') = \mathbb{S}/(\pi M \cap \pi M') \oplus \rho(M + M')/(\rho M + \rho M')$$

Proof. Apply Proposition 7.3.14 and Proposition 7.3.15. □

Equivalently, we have:

$$(\pi M \cap \pi M')/\pi(M \cap M') = \rho(M + M')/(\rho M + \rho M')$$

As a list of corollaries, we get:

Corollary 7.4.9. *If $(s, r) \in M$ and $(s, r') \in M'$, then: $s \in \pi(M \cap M')$ if, and only if, $r - r' \in \rho M + \rho M'$.*

Proof. If $s \in \pi(M \cap M')$, then there exists an $(s, r^*) \in M \cap M'$. We then get $r - r^* \in \rho(M)$ and $r^* - r' \in \rho(M')$ and thus $r - r' \in \rho(M) + \rho(M')$. Conversely, if $r - r' \in \rho(M) + \rho(M')$, then $r - r' = w - w'$ with $w \in \rho(M)$ and $w' \in \rho(M')$. Thus $(0, w) \in M$ and $(0, w') \in M'$ and we get $(s, r - w) = (s, r' - w') \in M \cap M'$. □

Corollary 7.4.10. *We have $\pi(M \cap M') = \pi M \cap \pi M'$ if, and only if, $\rho(M + M') = \rho M + \rho M'$.*

Proof. We have $(\pi M \cap \pi M')/\pi(M \cap M') = \rho(M + M')/(\rho M + \rho M')$. □

Corollary 7.4.11. *If $\rho M + \rho M' = \mathbb{R}$, then $\pi(M \cap M') = \pi M \cap \pi M'$.*

Proof. If $\rho M + \rho M' = \mathbb{R}$, then $\rho(M + M')/(\rho M + \rho M') = 0$. □

Corollary 7.4.12. *If $\rho(M + M') = 0$, then $\pi(M \cap M') = \pi M \cap \pi M'$.*

Proof. If $\rho(M + M') = 0$, then $\rho(M + M')/(\rho M + \rho M') = 0$. □

Remark Suppose $\pi(M)$ is interpreted (or labeled) as the output of the system M. To recover $\rho(M)$, we then force the output to be 0 and collect the allowable outcomes. As such $\rho(M)$ can be seen to correspond to the *zero-dynamics* of the system (see, e.g., [Isi95] Ch 4, Sec 4.3 and [Isi13]) interpreted through the lens of the behavioral approach. The link to zero-dynamics and its potential implications are not pursued in this paper, but may be a subject of further investigation. The same interpretation will reappear in the subsequent classes of systems, but will not be explicitly stated. Work, beyond the scope of this paper, is however needed to bring the link to fruition.

7.4.3 Localized changes

Any change is itself a system in $\mathbb{M}$. In case the change is *localized*, we get a stronger characterization. Suppose that the change will only be incurred on a subsystem of M. Specifically, we have a surjection $p : \mathbb{M} \to \mathbb{S}_c$. As the spaces in concern are vector spaces, the map p induces a linear inclusion map $i : \mathbb{S}_c \to \mathbb{M}$ such that pi is the identity on $\mathbb{S}_c$. The map ip is definitely not the identity map on $\mathbb{M}$.

Definition 7.4.13. *A change* $(\mathbb{M}, C)$ *is said to be localized to* $\mathbb{S}_c$ *if* $ip(C) = C$.

In such a setting, incurring the change C into M consists, equivalently, of interconnecting the system $(\mathbb{M}, M)$ with the system $(\mathcal{S}_c, p(C))$, living in a different universum. Restricting C to be a system on $\mathcal{S}_c$ yields us some gain in the characterization.

Let $(\mathbb{M}, C)$ be localized to $\mathbb{S}_c$. The following sequence is then exact:

$$0 \to \mathbb{M}/(M \cap ip(C)) \to \mathbb{M}/M \oplus \mathbb{S}_c/p(C) \to \mathbb{S}_c/(p(M) + p(C)) \to 0.$$

We can then construct a commutative ladder diagram, with exact rows, whose rightmost column is:

$$\begin{tikzcd}
\mathbb{S}_c \arrow[d] \\
\mathbb{S} \oplus \mathbb{S}_c/(p(M) + p(C))
\end{tikzcd}$$

The map $\mathbb{S}_c \to \mathbb{S}$ is πi. Furthermore, note that:

$$\mathbb{S}_c/(pM + pC) = \mathbb{M}/(ipM + ipC).$$

Let $\iota : \mathbb{R} \to \mathbb{M}$ be the canonical linear inclusion such that $\rho\iota = id_{\mathbb{R}}$. Then applying Proposition 7.3.14 (namely the snake lemma) to our diagram yields:

Proposition 7.4.14. *If* C *is a subspace of* $\mathbb{M}$ *be localized to* $\mathbb{S}_c$*, then:*

$$(\pi M \cap \pi C)/\pi(M \cap C) = p\iota\rho(ipM + ipC)/(p\iota\rho M + p\iota\rho C).$$

We also have $p\iota\rho(M + C) = p\iota\rho i(pM + pC)$.

Proof. All the needed elements of the proof have been described leading up to the proposition. The last statement is due to $C = p(C) \oplus \mathbb{R}_c$. Then $M + C = M + (p(C) \oplus \mathbb{R}_c) = (p(M) + p(C)) \oplus \mathbb{R}_c$. □

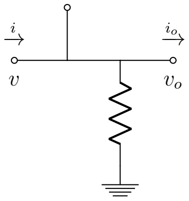
The spaces $p\iota\rho(ipM + ipC)$ and $(p\iota\rho M + p\iota\rho C)$ are both subspaces of $\mathbb{S}_c$. Thus if $\mathbb{S}_c$ is a very small space, we acquire a good computational gain. We can also further optimize the left hand side $(\pi M \cap \pi C)/\pi(M \cap C)$, but the intersection cannot be performed on a subspace smaller than $\mathbb{S}_c + \mathbb{S}$, if both are viewed as subspaces of $\mathbb{M}$. A corollary then becomes:

Corollary 7.4.15. *If $(s, r) \in M$ and $(s, r') \in C$ then:*

$$s \in \pi(M \cap M') \text{ if, and only if, } p\iota r - p\iota r' \in p\iota\rho M + p\iota\rho C.$$

Proof. The proof goes long the same lines as that of Corollary 7.4.9. □

As a concrete instantiation of Corollary 7.4.15 (and 7.4.9) , we consider the following *open* circuit:

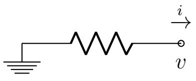


Let us suppose that the universum is $\mathbb{R}^4$ (the 4-dimensional vector space over the reals) corresponding to tuples (v, i, v_o, i_o). We denote the behavior of the circuit by $M \subseteq \mathbb{R}^4$. The behavior M corresponds to the tuples (v, i, v_o, i_o) that satisfy the laws of the circuit. The subsystem in concern corresponds to the output terminal, namely the pair (v_o, i_o). The behavior of the subsystem is then $\pi(M)$. The space $\rho(M)$ in this example will consist of the pairs (v, i) satisfying the circuit laws if i_0 and v_o are forced to be 0. Thus $\rho(M) = \{(0, i) : i \in \mathbb{R}\}$. Indeed, the non-labeled terminal may admit a non-zero current, as the circuit is *open* (i.e., free to interact with other circuits). Suppose we choose an output (V_o, I_o), the question is:

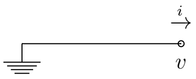
Question. *Can such an output be possible if we* append *(or glue) a circuit C at the left terminal?*

Let $p(C)$ denote the projection of C onto the terminal (v, i). We deduce from Corollary 7.4.9 that the answer to the question is **yes** if, and only if, there is a non-zero voltage v that is possible in $p(C)$. Explicitly, we know that $(V_0, i, V_0, I_0) \in M$. The answer is then yes if, and only if, $(V_0, i) - 0 \in p(C) + \rho(M)$. As $\rho(M) = \{(0, i) : i \in \mathbb{R}\}$, the answer is equivalent to $p(C)$ containing a pair with a non-zero voltage.

For instance, in the case where C is:



The answer is yes; the pair (V_o, I_o) can be made to appear at the subsystem. However, in the case where C is:



The answer is no; we can never observe (V_0, I_0) at the subsystem.

Further characterizations may be further deduced, but we move on to other classes of systems. We nevertheless mention an alternative lift that could be performed.

7.4.4 An alternative lift

Instead of lifting our situation through L, we could have performed an alternative lift. Let us associate to every subspace M of $\mathbb{M}$ a linear map:

$$g_M := \begin{tikzcd} M \arrow[d, "\pi i"] \\ \mathbb{S} \end{tikzcd}$$

i.e., the composition $M \xrightarrow{i} \mathbb{M} \xrightarrow{\pi} \mathbb{S}$, and to every order relation $M \subseteq M'$, or equivalently inclusion map $M \to M'$ to a morphism $g_M \Rightarrow g_{M'}$ of linear maps:

$$\begin{tikzcd} M \arrow[r, "\subseteq"] \arrow[d, "\pi i"'] & M' \arrow[d, "\pi i'"] \\ \mathbb{S} \arrow[r, "id"'] & \mathbb{S} \end{tikzcd}$$

It follows that $\operatorname{coker} g_M = \mathbb{S}/\pi M$ and $\ker g_M = \rho(M)$. Furthermore, for every M and M', the sequence $0 \Rightarrow g_{M\cap M'} \Rightarrow g_M \oplus g_{M'} \Rightarrow g_{M+M'} \Rightarrow 0$, is exact. Finally, cascade-like phenomena also occur precisely when there is loss of exactness. Indeed, for every M and M', we have $\pi(M \cap M') \neq \pi(M) \cap \pi(M')$ if, and only if, the sequence $0 \to \operatorname{coker} g_{M\cap M'} \to \operatorname{coker} g_M \oplus \operatorname{coker} g_{M'} \to \operatorname{coker} g_{M+M'} \to 0$ is not exact at $\operatorname{coker} g_{M\cap M'}$. Thus, for all purposes, we could have recovered the exact characterizations in this section via this alternative lift.

7.5 Affine systems over fields

Let $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{R}$ be finite dimensional vector spaces. For ease of presentation, we will fix a basis for $\mathbb{M}$ denoted by $\{s_1, \cdots, s_m, r_1, \cdots, r_n\}$ consistent with the factorization, namely such that $\{s_1, \cdots, s_m\}$ is a basis for $\mathbb{S}$ and $\{r_1, \cdots, r_n\}$ is a basis for $\mathbb{R}$. An element of $\mathbb{M}$ can be seen to correspond to a pair (s, r) where s and r are a linear combinations of the s_i's and r_i's, respectively.

Definition 7.5.1. *A Willems system $(\mathbb{U}, \mathcal{B})$ is said to be k-affine if $\mathbb{U}$ is a k-vector space, and $\mathcal{B}$ is an affine subspace of $\mathbb{U}$. The system $(\mathbb{U}, \mathcal{B})$ is additionally said to be finite-dimensional if $\mathbb{U}$ has finite dimension.*

A system $(\mathbb{M}, M)$ is then said to be k-affine if M is the set of solutions (in m) to $Am = b$ for some matrix A and some vector b. In case b is the zero vector, we recover the linear case. Thus M is typically defined by a set of equations $\{a_i^T s + b_i^T r = c_i\}_i$.

The veil $\pi : \mathbb{M} \to \mathbb{S}$ is a surjective map that projects an element $m = (s, r) \in \mathbb{M}$ into an element $s \in \mathbb{S}$.

7.5.1 The space of equations

Let $(\mathbb{M}, M)$ be a system defined by equations $\{a_i^T s + b_i^T r = c_i\}_i$. To each subspace M, we assign $\mathcal{I}(M)$ to be the set of annihilators of M. Precisely we let $\mathbb{I}_\mathbb{M}$ be the free k-vector space generated by the basis $\{1, s_1, \cdots, s_m, r_1, \cdots, r_n\}$. This basis contains 1, s_i's and r_i's as formal variables. Each equation $a_i^T s + b_i^T r = c_i$ can be regarded as an element of $\mathbb{I}_\mathbb{M}$. We can thus interpret $\mathbb{I}_\mathbb{M}$ as the k-vector space of equations, and by equation $a_i^T s + b_i^T r = c_i$ we will mean the tuple of its coefficients in $\mathbb{I}_\mathbb{M}$.

Definition 7.5.2. *For every subset M of $\mathbb{M}$ (non-necessarily a subspace) we define $\mathcal{I}(M)$ to be the smallest subspace of $\mathbb{I}_\mathbb{M}$ generated by the equations $\{a_i^T s + b_i^T r = c_i\}$ that are satisfied by all points of M.*

Proposition 7.5.3. *For every $M, M' \subseteq \mathbb{M}$, we have $\mathcal{I}(M \cup M') = \mathcal{I}(M) \cap \mathcal{I}(M')$.*

Proof. Trivially, $M \cup M'$ satisfies an affine equation e if, and only if, both M and M' satisfy e. □

Dually, for every subset I of $\mathbb{I}_\mathbb{M}$, we define $\mathcal{V}(I)$ to be the solution set in $\mathbb{M}$ to the equations in I.

Proposition 7.5.4. *The system $(\mathbb{M}, M)$ is a k-affine system if, and only if, $M = \mathcal{V}\mathcal{I}(M)$.*

Proof. The operator $\mathcal{V}\mathcal{I}$ can be easily shown to be closure operator on the lattice of subsets of $\mathbb{M}$. It sends a subset of $\mathbb{M}$ to the smallest affine subspace containing it. Its fixed-points are then the k-affine subspaces of $\mathbb{M}$. See also [Har13] Proposition 1.2, (e) for a similar statement. □

Thus knowing $\mathcal{I}(M)$, we can recover M whenever it is an affine subspace. Moreover, if $\mathcal{I}(M) = \mathcal{I}(M')$, then $M = M'$ (whenever M and M' are both affine subspaces, of course).

Proposition 7.5.5. *If M is the solution to the set of equations $E := \{a_i^T s + b_i^T r = c_i\}$, then $\mathcal{I}(M) = \langle E \rangle$.*

Proof. Clearly $\langle E \rangle \subseteq \mathcal{I}(M)$. Suppose $\langle E \rangle$ is a proper subspace of $\mathcal{I}(M)$, then $\dim \mathcal{I}(M) > \dim \langle E \rangle$, and so $\mathcal{V}\mathcal{I}(M) < \dim \mathcal{V}\langle E \rangle$. But by Proposition 7.5.4, we have $\mathcal{V}\mathcal{I}(M) = M = \mathcal{V}\langle E \rangle$, contradicting the fact that $\langle E \rangle$ is a proper subspace of $\mathcal{I}(M)$. □

We then get:

Corollary 7.5.6. *If M and M' are affine subspaces, then $\mathcal{I}(M \cap M') = \mathcal{I}(M) + \mathcal{I}(M')$.*

Proof. We have $\mathcal{V}(\mathcal{I}(M) + \mathcal{I}(M')) = M \cap M'$. The rest then follows by Proposition 7.5.5. □

The situation is a special case of that of systems defined by polynomial equations. All the polynomials in the affine case have degree one. We thus trivially get more mathematical structure and a greater flexibility to work with.

We similarly define $\mathbb{I}_{\mathbb{S}}$ to be the free vector space generated by $\{1, s_1, \cdots, s_m\}$. The projection $\pi : \mathbb{M} \to \mathbb{S}$ then induces an inclusion $\iota : \mathbb{I}_{\mathbb{S}} \to \mathbb{I}_{\mathbb{M}}$. The operator $\mathcal{I}$ can act on the subsets of $\mathbb{S}$, and the operator $\mathcal{V}$ act on the subsets of $\mathbb{I}_{\mathbb{S}}$. In such a case, $(\mathbb{S}, S)$ is also k-affine if and only if $S = \mathcal{V}\mathcal{I}(S)$. We can then uniquely recover S knowing $\mathcal{I}(S)$. The space $\mathbb{I}_{\mathbb{S}}$ defines a subspace of $\mathbb{I}_{\mathbb{M}}$.

7.5.2 The lift to linear maps

Let $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{R}$ be finite dimensional vector spaces. Our behaviors will be affine subspaces of their respective universa. For a k-affine system $(\mathbb{M}, M)$, we then define the following lift:

Definition 7.5.7. *We define L to be a functor that sends every affine subspace M to:*

$$h_M := \begin{tikzcd} \mathbb{I}_{\mathbb{S}} \arrow[d, "p\iota"] \\ \mathbb{I}_{\mathbb{M}}/\mathcal{I}(M) \end{tikzcd}$$

i.e., the composition $\mathbb{I}_{\mathbb{S}} \xrightarrow{\iota} \mathbb{I}_{\mathbb{M}} \xrightarrow{p} \mathbb{I}_{\mathbb{M}}/\mathcal{I}(M)$, and to every order relation $M \subseteq M'$, or equivalently inclusion map $M \to M'$ to a morphism $h_{M'} \Rightarrow h_M$ of linear maps:

$$\begin{tikzcd} \mathbb{I}_{\mathbb{S}} \arrow[r, "id"] \arrow[d, "p\iota"'] & \mathbb{I}_{\mathbb{S}} \arrow[d, "p\iota"] \\ \mathbb{I}_{\mathbb{M}}/\mathcal{I}(M') \arrow[r] & \mathbb{I}_{\mathbb{M}}/\mathcal{I}(M) \end{tikzcd}$$

Note that the inclusion $M \subseteq M'$ becomes a morphism $h_{M'} \Rightarrow h_M$ in the opposite direction. The roles of the kernel and the cokernel will then be reversed. The lift encodes the behavior of the subsystem in the kernel:

Proposition 7.5.8. *For every M, we have* $\ker h_M = \mathcal{I}(M) \cap \mathbb{I}_{\mathbb{S}}$.

Proof. We have $e \in \ker p\iota$ if, and only if, $e \in \mathbb{I}_{\mathbb{S}}$ and $\iota e \in \mathcal{I}(M)$. □

The kernel then encodes the behavior of the subsystem:

Proposition 7.5.9. *For every M, we have that* $\mathcal{V}(\ker h_M) \cap \mathbb{S}$ *is* $\pi(M)$.

Proof. We have $\mathbb{I}_{\mathbb{S}} = \mathcal{I}(\mathbb{R})$. Then $\mathcal{I}(M) \cap \mathbb{I}_{\mathbb{S}} = \mathcal{I}(M) \cap \mathcal{I}(\mathbb{R}) = \mathcal{I}(M \cup \mathbb{R})$. But $\mathcal{V}\mathcal{I}(M \cup \mathbb{R}) = \pi(M) \oplus \mathbb{R}$ as a subspace of $\mathbb{I}_{\mathbb{M}}$. □

The lift also preserves construction, or interconnection, of systems. It however flips the direction of the morphisms.

Proposition 7.5.10. *The sequence* $0 \Rightarrow h_{M+M'} \Rightarrow h_M \oplus h_{M'} \Rightarrow h_{M \cap M'} \Rightarrow 0$ *is exact, for every M and M'.*

Proof. The sequence:

$$0 \to \mathbb{I}_{\mathbb{M}}/(\mathcal{I}(M+M')) \to \mathbb{I}_{\mathbb{M}}/(\mathcal{I}(M)) \oplus \mathbb{I}_{\mathbb{M}}/(\mathcal{I}(M')) \to \mathbb{I}_{\mathbb{M}}/(\mathcal{I}(M \cap M')) \to 0$$

is exact. Indeed, for all affine subspaces M and M', we have $\mathcal{I}(M \cup M') = \mathcal{I}(M) \cap \mathcal{I}(M')$ and $\mathcal{I}(M \cup M') = \mathcal{I}(M + M')$. We also have $\mathcal{I}(M \cap M') = \mathcal{I}(M) + \mathcal{I}(M')$ (by 7.5.6). □

Finally, cascade-like phenomena occur precisely when there is loss of exactness. The loss of exactness is, however, on the right:

Proposition 7.5.11. *For every M and M', we have $\pi(M \cap M') \neq \pi(M) \cap \pi(M')$ if, and only if, the sequence $0 \to \ker h_{M+M'} \to \ker h_M \oplus \ker h_{M'} \to \ker h_{M \cap M'} \to 0$ is not exact at $\ker h_{M \cap M'}$.*

Proof. We have $\mathcal{V}\big((\mathcal{I}(M)+\mathcal{I}(M')) \cap \mathbb{I}_{\mathbb{S}}\big) = \pi(M \cap M')$ as $\mathcal{I}(M)+\mathcal{I}(M') = \mathcal{I}(M \cap M')$. We also have $\mathcal{V}(\mathcal{I}(M) \cap \mathbb{I}_{\mathbb{S}}) = \pi(M)$. The result then follows as:

$$(\mathcal{I}(M) + \mathcal{I}(M')) \cap \mathbb{I}_{\mathbb{S}} \neq (\mathcal{I}(M) \cap \mathbb{I}_{\mathbb{S}}) + (\mathcal{I}(M') \cap \mathbb{I}_{\mathbb{S}})$$

if, and only if $\pi(M \cap M') \neq \pi(M) \cap \pi(M')$. □

The object encoding the potential for this loss is the cokernel.

Proposition 7.5.12. *We have* $\operatorname{coker} h_M = \mathbb{I}_{\mathbb{M}}/(\mathcal{I}(M) + \mathbb{I}_{\mathbb{S}})$.

Proof. The image of $\pi\iota$ is $(\mathcal{I}(M) + \mathbb{I}_{\mathbb{S}})/\mathcal{I}(M)$. The result then follows from the third isomorphism theorem. □

The cokernel represents the projection of the equations in play. Note that as all equations in $\operatorname{coker} h_M$ are linear, they can be identified with a linear subspace of $\mathbb{R}$.

7.5.3 Recovering exactness

To recover exactness, we let $\rho(M)$ be $\mathcal{I}(M) + \mathbb{I}_{\mathbb{S}}$.

Definition 7.5.13. *We define $\rho(M)$ to be the set of equations $\{b_i^T r = 0 : a_i^T s + b_i^T r = c_i$ is in $\mathcal{I}(M)\}$.*

We then obtain a characterization of the form:

Theorem 7.5.14. *We have:*

$$\mathcal{I}\pi(M \cap M') = (\mathcal{I}\pi M + \mathcal{I}\pi M') \oplus \rho(M) \cap \rho(M')/\rho(M + M')$$

Proof. Combine Proposition 7.3.14 and Proposition 7.3.15. □

It is possible to construct a generating set for $\mathcal{I}\pi(M \cap M')$ using the characterization. Suppose we have a generating set B_M, B'_M and B_{common} for $\mathcal{I}\pi M$, $\mathcal{I}\pi M'$ and $\rho(M) \cap \rho(M')/\rho(M + M')$, respectively. For every element in B_{common} retrieve a preimage e in $\mathcal{I}(M)$ and a preimage e' in $\mathcal{I}(M')$, with respect to ρ. The set containing B_M, B'_M and the equation $e - e'$ for every element of B_{common} will constitute a basis for $\mathcal{I}\pi(M \cap M')$.

Link to the linear case If we set all the constants c_i in the equations to 0, we recover the linear case. Thus the dual process of defining a space of equations (or equivalently the space of annihilators) gives us a means to recover a generating set to the linear case too.

We also know that an affine subspace consists of a linear subspace whose every element is offset by a fixed vector. If we know a point s in $\pi(M + M')$, we can force all offsets, or constants c_i to be zero, then solve the problem as done in the linear case. Every solution to the affine case, is then a solution to the linear case offset by s.

7.6 Finite systems

The behaviors in this section will consist of finite sets, without any equipped structure.

Definition 7.6.1. *A Willems system $(\mathbb{U}, \mathcal{B})$ is said to be finite if $\mathbb{U}$ is a finite set, and $\mathcal{B}$ is a subset of $\mathbb{U}$.*

Let $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{R}$ be finite sets. The veil $\pi : \mathbb{M} \to \mathbb{S}$ is a surjective map that projects an element $m = (s, r) \in \mathbb{M}$ into an element $s \in \mathbb{S}$. Sets do not possess a linear structure; the goal is to first lift our situation to become linear. This can be achieved by encoding the elements of a set into the dimensions of a free module. As our sets are finite, this encoding yields a finite dimensional objects.

7.6.1 Free constructions

Let $\mathcal{R}$ be any commutative ring with unit.

Definition 7.6.2. *Given a set S, we denote by $\mathcal{R}^S$ the free $\mathcal{R}$-module generated by the elements of S.*

An element of $\mathcal{R}^S$ can be formally thought of as a set map from S to $\mathcal{R}$. Each dimension encodes an element of S. The set maps can be added together and multiplied by elements of $\mathcal{R}$. If $S' \subseteq S$, then we denote by $\langle S' \rangle$ the linear subspace of $\mathcal{R}^S$ spanned by the elements of S'. The space $\langle S' \rangle$ is then isomorphic to $\mathcal{R}^{S'}$. Finally, the free construction is functorial in the following sense:

Proposition 7.6.3. *Every map $f : S \to T$ lifts to a linear map $\bar{f} : \mathcal{R}^S \to \mathcal{R}^T$ on free $\mathcal{R}$-modules.*

Proof. For $s, s' \in S$, define $\bar{f}(\alpha s + \alpha' s') = \alpha f(s) + \alpha' f(s')$ whenever $\alpha, \alpha' \in \mathcal{R}$. □

Specifically, the projection $\pi : \mathbb{M} \to \mathbb{S}$ lifts to a linear map $\Pi : \mathcal{R}^{\mathbb{M}} \to \mathcal{R}^{\mathbb{S}}$.

7.6.2 The lift to linear maps

We perform the following lift:

Definition 7.6.4. *We define L to be a functor that sends every subset M of $\mathbb{M}$ to:*

$$h_M := \begin{tikzcd} \mathcal{R}^{\mathbb{M}} \arrow[d, "{(\Pi,p)}"] \\ \mathcal{R}^{\mathbb{S}} \oplus (\mathcal{R}^{\mathbb{M}}/\langle M \rangle) \end{tikzcd}$$

and every order relation $M \subseteq M'$, or equivalently inclusion map $M \to M'$ to a morphism $h_M \Rightarrow h_{M'}$ of linear maps:

$$\begin{tikzcd} \mathcal{R}^{\mathbb{M}} \arrow[r, "id"] \arrow[d, "{(\Pi,p)}"'] & \mathcal{R}^{\mathbb{M}} \arrow[d, "{(\Pi,p')}"] \\ \mathcal{R}^{\mathbb{S}} \oplus (\mathcal{R}^{\mathbb{M}}/\langle M \rangle) \arrow[r, "{(id,i)}"'] & \mathcal{R}^{\mathbb{S}} \oplus (\mathcal{R}^{\mathbb{M}}/\langle M' \rangle) \end{tikzcd}$$

The behavior of the subsystem is again encoded in the cokernel:

Proposition 7.6.5. *For every M, we have* $\operatorname{coker} h_M = \mathcal{R}^{\mathbb{S}}/\Pi(\langle M \rangle)$.

Proof. The image of (Π, p) is $\Pi(\langle M \rangle) \oplus \mathcal{R}^{\mathbb{M}}/\langle M \rangle$. □

The lift also preserves the interconnection of systems:

Proposition 7.6.6. *The sequence $0 \Rightarrow h_{M \cap M'} \Rightarrow h_M \oplus h_{M'} \Rightarrow h_{M \cup M'} \Rightarrow 0$ is exact, for every M and M'.*

Proof. We have $\langle M \cap M' \rangle = \langle M \rangle \cap \langle M' \rangle$ and $\langle M \cup M' \rangle = \langle M \rangle + \langle M' \rangle$. The sequence $0 \to \mathcal{R}^{\mathbb{M}}/\langle M \cap M' \rangle \to \mathcal{R}^{\mathbb{M}}/\langle M \rangle \oplus \mathcal{R}^{\mathbb{M}}/\langle M' \rangle \to \mathcal{R}^{\mathbb{M}}/\langle M + M' \rangle \to 0$ is then exact. Direct sum of exact sequences yields an exact sequence. The two rows of the obtained ladder diagram are then easily seen to be exact. □

Finally, cascade-like phenomena occur precisely when there is loss of exactness:

Proposition 7.6.7. *For every M and M', we have $\pi(M \cap M') \neq \pi(M) \cap \pi(M')$ if, and only if, the sequence $0 \to \operatorname{coker} h_{M \cap M'} \to \operatorname{coker} h_M \oplus \operatorname{coker} h_{M'} \to \operatorname{coker} h_{M+M'} \to 0$ is not exact at $\operatorname{coker} h_{M \cap M'}$.*

Proof. We have $\operatorname{coker} h_{M \cap M'} = \mathcal{R}^{\mathbb{S}}/\Pi\langle M \cap M' \rangle$. Exactness on the left is lost if, and only if, $\Pi\langle M \cap M' \rangle \neq \Pi\langle M \rangle \cap \Pi\langle M' \rangle$. □

7.6.3 Recovering Exactness

We define ρ to be the map sending M into $\rho(M) = \ker h_M$.

Definition 7.6.8. *We define $\rho : M \mapsto \langle\{(s,r) - (s,r') \in \mathcal{R}^{\mathbb{M}} : (s,r),(s,r') \in M\}\rangle$.*

Thus, if for every s there exists at most one $(s,r) \in M$, then $\rho(M)$ is the 0 subspace of $\mathcal{R}^{\mathbb{M}}$. One way to interpret $\rho(M)$ is as a map from $\mathbb{S}$ to the subspaces of $\mathcal{R}^{\mathbb{R}}$. The module $\rho(M)$ is then isomorphic to the direct sum of the images. We again directly obtain a characterization of the form:

Theorem 7.6.9. *For every M and M', we have:*

$$(\langle\pi M\rangle \cap \langle\pi M'\rangle)/\langle\pi(M \cap M')\rangle = \rho(M \cup M')/(\rho M + \rho M')$$

Proof. Apply Proposition 7.3.14 and Proposition 7.3.15. □

Note that in this case, the space $(\pi\langle M\rangle \cap \pi\langle M'\rangle)/\pi(\langle M\rangle \cap \langle M'\rangle)$ is freely generated by the element of the set $\pi(M) \cap \pi(M') - \pi(M \cap M')$.

The single set problem The same problem admits a formalization of a different kind. Suppose $\mathbb{S}$ consists of a single element. Recovering $\pi(M \cap M')$ is equivalent to dually asking whether $(\mathbb{M} - M') \cup (\mathbb{M} - M')$ is equal to $\mathbb{M}$ or not. One can get an answer to such a dual question through the following lift. The lift sends M to $\langle m_1 + \cdots + m_n\rangle \to \mathcal{R}^{\mathbb{M}}/\langle M\rangle$. The kernel is $\mathcal{R}$ if, and only if, the set M is equal to $\mathbb{M}$.

7.7 Linear time-invariant systems

Let us fix a field k throughout this section. We define $\mathcal{R}$ to be the ring $k[[z^{-1}]]$ of formal power series in the variable z^{-1} with coefficients in k. The elements in $\mathcal{R}$ are series of the form $\sum_{i=0}^{\infty} a_i z^{-i}$ with $a_i \in k$. Such an element ought to be interpreted as a discrete-time signal taking value a_i in k at time i. Addition in $\mathcal{R}$ is given by pointwise addition $\sum_i a_i z^{-i} + \sum_i b_i z^{-i} = \sum_i (a_i + b_i) z^{-i}$, and multiplication is given by convolution $(\sum_i a_i z^{-i})(\sum_i b_i z^{-i}) = \sum_i (a_0 b_i + \cdots + a_i b_0) z^{-i}$.

Definition 7.7.1. *A Willems system $(\mathbb{U}, \mathcal{B})$ is said to be a linear time-invariant (LTI) system if $\mathbb{U}$ is a free $\mathcal{R}$-module $\mathcal{R}^n$ of dimension n, and $\mathcal{B}$ is an $\mathcal{R}$-submodule of $\mathbb{U}$.*

The system $(\mathbb{U}, \mathcal{B})$ is *linear* as $\alpha s + \alpha' s' \in \mathcal{B}$ for every $\alpha, \alpha' \in k$ and $s, s' \in \mathcal{B}$. The system $(\mathbb{U}, \mathcal{B})$ is *time-invariant* as $z^{-1}s \in \mathcal{B}$ for every $s \in \mathcal{B}$.

7.7.1 A lift to linear maps

Let $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{R}$ be finite-dimensional free $\mathcal{R}$-modules. The veil $\pi : \mathbb{M} \to \mathbb{S}$ is a surjective $\mathcal{R}$-linear map that projects an element $m = (s,r) \in \mathbb{M}$ into an element $s \in \mathbb{S}$.

Definition 7.7.2. *We define L to be a functor that sends every submodule M of $\mathbb{M}$ to:*

$$h_M := \begin{tikzcd} \mathbb{M} \arrow[d, "{(\pi,p)}"] \\ \mathbb{S} \oplus (\mathbb{M}/M) \end{tikzcd}$$

and every order relation $M \subseteq M'$, or equivalently inclusion map $M \to M'$, to a morphism $h_M \Rightarrow h_{M'}$ of linear maps:

$$\begin{tikzcd} \mathbb{M} \arrow[r, "id"] \arrow[d, "{(\pi,p)}"'] & \mathbb{M} \arrow[d, "{(\pi,p')}"] \\ \mathbb{S} \oplus (\mathbb{M}/M) \arrow[r, "{(\mathrm{id},i)}"'] & \mathbb{S} \oplus (\mathbb{M}/M') \end{tikzcd}$$

The lift encodes the behavior of the subsystem in the cokernel of the map:

Proposition 7.7.3. *For every M, we have $\operatorname{coker} h_M = \mathbb{S}/\pi(M)$.*

Proof. The image of (π, p) is $\pi(M) \oplus \mathbb{M}/M$. □

Proposition 7.7.4. *The sequence $0 \Rightarrow h_{M\cap M'} \Rightarrow h_M \oplus h_{M'} \Rightarrow h_{M+M'} \Rightarrow 0$ is exact, for every M and M'.*

Proof. The proof is similar to that of the case of linear systems over fields. The sequence $0 \to \mathbb{M}/(M \cap M') \to \mathbb{M}/M \oplus \mathbb{M}/M' \to \mathbb{M}/(M + M') \to 0$ is exact. Direct sum of exact sequences yields an exact sequence. The two rows of the obtained ladder diagram are then easily seen to be exact. □

Finally, cascade-like phenomena occur precisely when there is loss of exactness:

Proposition 7.7.5. *For every M and M', we have $\pi(M\cap M') \neq \pi(M)\cap\pi(M')$ if, and only if, the sequence $0 \to \operatorname{coker} h_{M\cap M'} \to \operatorname{coker} h_M \oplus \operatorname{coker} h_{M'} \to \operatorname{coker} h_{M+M'} \to 0$ is not exact at $\operatorname{coker} h_{M\cap M'}$.*

Proof. We have $\operatorname{coker} h_{M\cap M'} = \mathbb{S}/\pi(McapM')$. Exactness on the right is then lost if, and only if, $\pi(M \cap M') \neq \pi(M) \cap \pi(M')$. □

The object encoding the potential for this loss is the kernel.

Proposition 7.7.6. *We have $\ker h_M = \{r \in \mathbb{R} : (0, r) \in M\}$ is a submodule of $\mathbb{R}$.*

Proof. We have $x \in \ker h_M$ if, and only if, $\pi(x) = 0$ and $x \in M$. □

We can then use this potential to recover the loss.

7.7.2 Recovering Exactness

Similarly, let us define ρ such that $M \mapsto \{r \in \mathbb{R} : (0, r) \in M\}$. We have $\rho(M)$ is $M \cap \{m : \pi(m) = 0\}$. We then get:

Theorem 7.7.7. *The following sequence is exact:*

$$0 \longrightarrow \rho(M + M)/\rho(M) + \rho(M') \longrightarrow \mathbb{S}/\pi(M \cap M') \longrightarrow \mathbb{S}/\pi(M) \cap \pi(M') \longrightarrow 0$$

Proof. Apply Proposition 7.3.14 and Proposition 7.3.16. □

The sequence however does not necessarily *split* as in Proposition 7.3.15. We cannot thus get a direct sum characterization as that in the previous three sections. Indeed, consider the following exact sequence:

$$0 \to \mathcal{R}/z^{-1}\mathcal{R} \to M \to \mathcal{R}/z^{-1}\mathcal{R} \to 0.$$

Without any additional information, the module M can be either $\mathcal{R}/z^{-1}\mathcal{R} \oplus \mathcal{R}/z^{-1}\mathcal{R}$ or $\mathcal{R}/z^{-2}\mathcal{R}$. The required information, to know which one it is, is encoded in the maps. We however do not know the maps. To overcome such an ambiguity, let us suppose that we *know* the module $M_{z^{-1}} = M \otimes \mathcal{R}/z^{-1}\mathcal{R}$. If $M_{z^{-1}}$ is $(\mathcal{R}/z^{-1}\mathcal{R})^2$, then M is $(\mathcal{R}/z^{-1}\mathcal{R})^2$. However, if $M_{z^{-1}}$ is $\mathcal{R}/z^{-1}\mathcal{R}$ then M is $\mathcal{R}/z^{-2}\mathcal{R}$.

Proposition 7.7.8. *Let p be prime in $\mathcal{R}$, and let N be a finitely generated $\mathcal{R}$-module. Given an exact sequence $0 \to M \to N \otimes \mathcal{R}/p^n\mathcal{R} \to P \to 0$ and (a presentation of) the module $N \otimes \mathcal{R}/p^{n-1}\mathcal{R}$, then $N \otimes \mathcal{R}/p^n\mathcal{R}$ is uniquely determined up to isomorphism.*

Proof. As $\mathcal{R}$ is a principal ideal domain, we know that N is isomorphic to a direct sum:

$$N = \mathcal{R}^m \oplus \mathcal{R}/(s_0\mathcal{R}) \oplus \cdots \oplus \mathcal{R}/(s_l\mathcal{R})$$

where $s_0, \cdots, s_l$ are powers of primes. (See e.g., [Lan02] Ch. III, Theorem 7.3 and Theorem 7.5 combined.) We also have:

$$\mathcal{R}/(s\mathcal{R}) \otimes \mathcal{R}/p^n\mathcal{R} = \mathcal{R}/(s\mathcal{R} + p^n\mathcal{R}) = \begin{cases} \mathcal{R}/p^n\mathcal{R} & \text{if } s = 0 \\ \mathcal{R}/p^{\min\{n,n'\}}\mathcal{R} & \text{if } s = p^{n'} \\ 0 & \text{otherwise} \end{cases}$$

Without loss of generality, we may then assume N to be isomorphic to:

$$N = \mathcal{R}^m \bigoplus_{i>0} \left(\mathcal{R}/p^i\mathcal{R}\right)^{d_i}$$

for some non-negative integers $m, d_1, d_2, \cdots$ where finitely many are non-zero. We then have:

$$N \otimes \mathcal{R}/p^n\mathcal{R} = \left(\mathcal{R}/p\mathcal{R}\right)^{d_1} \oplus \cdots \oplus \left(\mathcal{R}/p^{n-1}\mathcal{R}\right)^{d_{n-1}} \oplus \left(\mathcal{R}/p^n\mathcal{R}\right)^{m+d_n+d_{n+1}+\cdots}$$

If A is a module and c is a non-negative integer, let $d(A, c)$ denote the largest d such that $(\mathcal{R}/p^c\mathcal{R})^d$ is isomorphic to a submodule of A. We then get the following set of equations:

$$\begin{aligned} d_i &= d(N \otimes \mathcal{R}/p^{n-1}\mathcal{R}, i) \quad \text{ for } i < n-1 \\ d_{n-1} + d_\infty &= d(N \otimes \mathcal{R}/p^{n-1}\mathcal{R}, n-1) \\ d_1 + 2d_2 + \cdots + (n-1)d_{n-1} + nd_\infty &= \sum_i id(M,i) + \sum_j jd(P,i) \end{aligned}$$

where $d_\infty = m + d_n + d_{n+1} + \cdots$. The solution to this set of equations yields the exponents needed to characterize $N \otimes \mathcal{R}/p^n\mathcal{R}$. □

This approach lends itself to a recursive characterization of the module $N \otimes \mathcal{R}/p^n\mathcal{R}$ when N is finitely generated. Regardless of the characterization, we still recover the following corollaries.

Corollary 7.7.9. *If $(s, r) \in M$ and $(s, r') \in M'$ then: $s \in \pi(M \cap M')$ if, and only if, $r - r' \in \rho M + \rho M'$.* □

Corollary 7.7.10. *We have $\pi(M \cap M') = \pi M \cap \pi M'$ if, and only if, $\rho(M + M') = \rho M + \rho M'$.* □

Corollary 7.7.11. *If $\rho M + \rho M' = \mathbb{R}$, then $\pi(M \cap M') = \pi M \cap \pi M'$.* □

Corollary 7.7.12. *If $\rho(M + M') = 0$, then $\pi(M \cap M') = \pi M \cap \pi M'$.* □

Deciding membership can usually be achieved through the use of Gröbner bases. Such a thread will not be pursued in this paper.

7.8 Algebraic systems

We fix an arbitrary field k. The field k does not have to be algebraically closed. Such a fact will not have any implications on the development in this section.

Definition 7.8.1. *A Willems system $(\mathbb{U}, \mathcal{B})$ is said to be an algebraic system over k if $\mathbb{U}$ is the affine n-space k^n and $\mathcal{B}$ is an algebraic set in $\mathbb{U}$, namely the solution set to a collection of polynomial equations with coefficients in k.*

Of course, if M and M' are algebraic subsets of $\mathbb{M}$, then both $M \cap M'$ and $M \cup M'$ are algebraic sets.

As done in the affine case, we will fix a basis for $\mathbb{M}$ denoted by $\{s_1, \cdots, s_m, r_1, \cdots, r_n\}$ consistent with the factorization, namely such that $\{s_1, \cdots, s_m\}$ is a basis for $\mathbb{S}$ and $\{r_1, \cdots, r_n\}$ is a basis for $\mathbb{R}$. An element of $\mathbb{M}$ can be seen to correspond to a pair (s, r) where s and r are a linear combination of the s_i's and r_i's, respectively.

7.8.1 Polynomial ideals

We denote by $k[s_1, \cdots, s_m, r_1, \cdots, r_n]$ the ring of polynomials over the basis variables $\{s_1, \cdots, s_m, r_1, \cdots, r_n\}$ with coefficients in k. The ring $k[s_1, \cdots, s_m]$ is the polynomial ring over the variables $\{s_1, \cdots, s_m\}$ with coefficients in k. For ease of notation, we will denote $k[s_1, \cdots, s_m, r_1, \cdots, r_n]$ by $k[\boldsymbol{s}, \boldsymbol{r}]$ and $k[s_1, \cdots, s_m]$ by $k[\boldsymbol{s}]$.

The projection $\pi : \mathbb{M} \to \mathbb{S}$ induces an inclusion map of rings $\iota : k[\boldsymbol{s}] \to k[\boldsymbol{s}, \boldsymbol{r}]$. Given a subset $M \subseteq \mathbb{M}$, we define $\mathcal{I}(M)$ to be the ideal of $k[\boldsymbol{s}, \boldsymbol{r}]$ consisting of polynomials vanishing over M.

Proposition 7.8.2. *For every M and M' subsets of $\mathbb{M}$, we have $\mathcal{I}(M \cup M') = \mathcal{I}(M) \cap \mathcal{I}(M')$.*

Proof. Trivially, a polynomial vanishing at $M \cup M'$, then vanishes at M and at M'. See for instance [Har13], Proposition 1.2. □

Similarly, given a subset I of $k[\boldsymbol{s}, \boldsymbol{r}]$, we define the algebraic set $\mathcal{V}(I)$ to be the solution set of the set I of polynomials.

Proposition 7.8.3. *The system $(\mathbb{M}, M)$ is algebraic if, and only if, $\mathcal{V}\mathcal{I}(M) = M$.*

Proof. The operator $\mathcal{V}\mathcal{I}$ can be shown to be a closure operator on the lattice of subsets of $\mathbb{M}$. It sends a subset of $\mathbb{M}$ to the smallest algebraic containing it. Its fixed-points are then the algebraic subsets of $\mathbb{M}$. See also [Har13] Proposition 1.2, (e) for a supporting statement. □

Thus knowing $\mathcal{I}(M)$, we can recover M whenever it is an algebraic set.

Proposition 7.8.4. *If M is the solution to the set E of polynomial equations, then $\mathcal{V}(\langle E \rangle) = \mathcal{V}(\mathcal{I}(M)) = M$.*

Proof. We have $M = \mathcal{V}\mathcal{I}(M)$ by Proposition 7.8.3 and $M = \mathcal{V}(E) = \mathcal{V}(\langle E \rangle)$. □

However, it is not necessarily the case that $\langle E \rangle = \mathcal{I}(M)$. Such a fact will, however, not cause us any issues. Indeed:

Proposition 7.8.5. *For every subset I and I', we have $\mathcal{V}(I \cup I') = \mathcal{V}(I) \cap \mathcal{V}(I')$.*

Proof. Trivially, a point satisfies $I \cup I'$ if, and only if, it satisfies both I and I'. □

Corollary 7.8.6. *If M and M' are algebraic sets with $M = \mathcal{V}(I)$ and $M' = \mathcal{V}(I')$, then $\mathcal{V}(I \cup I') = M \cap M'$.* □

However, note that $\mathcal{V}(I \cap I')$ is non-necessarily $M \cup M'$.

7.8.2 A lift to linear maps

Let $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{R}$ be finite dimensional k-vector spaces. Our behaviors will be algebraic sets of their respective universa. The veil $\pi : \mathbb{M} \to \mathbb{S}$ is a surjective map that projects an element $m = (s, r) \in \mathbb{M}$ into an element $s \in \mathbb{S}$. The projection of an algebraic set however need not be an algebraic set. Thus, when we refer to $\pi(M)$, we will be referring the algebraic closure of the set $\pi(M)$.

We then define the following lift:

Definition 7.8.7. *We define L to be a functor that sends every algebraic subset M of $\mathbb{M}$ to:*

$$h_M := \begin{tikzcd} k[\boldsymbol{s}] \arrow[d, "\pi\iota"] \\ k[\boldsymbol{s},\boldsymbol{r}]/\mathcal{I}(M) \end{tikzcd}$$

i.e., the composition $k[\boldsymbol{s}] \xrightarrow{\iota} k[\boldsymbol{s},\boldsymbol{r}] \xrightarrow{\pi} k[\boldsymbol{s},\boldsymbol{r}]/\mathcal{I}(M)$, and to every order relation $M \subseteq M'$, or equivalently inclusion map $M \to M'$ to a morphism $h_{M'} \Rightarrow h_M$ of linear maps:

$$\begin{tikzcd} k[\boldsymbol{s}] \arrow[r, "id"] \arrow[d, "p'\iota"'] & k[\boldsymbol{s}] \arrow[d, "p\iota"] \\ k[\boldsymbol{s},\boldsymbol{r}]/\mathcal{I}(M') \arrow[r] & k[\boldsymbol{s},\boldsymbol{r}]/\mathcal{I}(M) \end{tikzcd}$$

The map h_M is linear in the following sense:

Proposition 7.8.8. *The module $k[\boldsymbol{s},\boldsymbol{r}]/\mathcal{I}(M)$ admits a $k[\boldsymbol{s}]$-module structure, and h_M can be regarded as a $k[\boldsymbol{s}]$-linear map of $k[\boldsymbol{s}]$-modules.*

Proof. The proof follows by restriction of scalars via $\iota : k[\boldsymbol{s}] \to k[\boldsymbol{s},\boldsymbol{r}]$. Indeed, define a $k[\boldsymbol{s}]$-action on the $k[\boldsymbol{s},\boldsymbol{r}]$-module M by $\alpha \cdot m = (\iota\alpha)m$ for $\alpha \in k[\boldsymbol{s}]$. The idea extends to the linear map. □

As in the affine case, the inclusion $M \subseteq M'$ becomes a morphism $h_{M'} \Rightarrow h_M$ in the opposite direction. The roles of the kernel and the cokernel will again be reversed. The lift then encodes the behavior of the subsystem in the kernel:

Proposition 7.8.9. *For every M, we have $\ker h_M = \mathcal{I}(M) \cap k[\boldsymbol{s}]$ where $\mathcal{I}(M)$ and $k[\boldsymbol{s}]$ are taken as $k[\boldsymbol{s}]$-modules.*

Proof. We have $e \in \ker p\iota$ if, and only if, $e \in k[\boldsymbol{s}]$ and $e \in \mathcal{I}(M)$. □

The set-projection of an algebraic set need not yield an algebraic set. However, as we are only working with algebraic systems, we would need to consider the algebraic closure of the projected set. Indeed:

Proposition 7.8.10. *For every algebraic subset M, we have that $\mathcal{V}(\ker h_M)$ (as a subset of $\mathbb{S}$) is the algebraic closure of $\pi(M)$.*

Proof. The statement is standard, see e.g., [CLO07] Ch 3 Section 2 Theorem 3. □

The lift also preserves construction, or interconnection, of systems, but in a slightly different sense. It however flips the direction of the morphisms.

Proposition 7.8.11. *The sequence $0 \Rightarrow h_{M\cup M'} \Rightarrow h_M \oplus h_{M'} \Rightarrow \big(k[\boldsymbol{s}] \to k[\boldsymbol{s},\boldsymbol{r}]/(\mathcal{I}M + \mathcal{I}M')\big) \Rightarrow 0$ is exact, for every M and M'.*

Proof. The sequence:

$$0 \to k[\boldsymbol{s}, \boldsymbol{r}]/\mathcal{I}(M \cup M') \to k[\boldsymbol{s}, \boldsymbol{r}]/\mathcal{I}(M) \oplus k[\boldsymbol{s}, \boldsymbol{r}]/\mathcal{I}(M') \to k[\boldsymbol{s}, \boldsymbol{r}]/(\mathcal{I}M + \mathcal{I}M') \to 0$$

is exact as $\mathcal{I}(M \cup M') = \mathcal{I}(M) \cap \mathcal{I}(M')$, see Proposition 7.8.2. □

In this case, note that $\mathcal{V}(\mathcal{I}M + \mathcal{I}M')$ is $M \cap M'$. Finally, cascade-like phenomena occur precisely when there is loss of exactness. The loss of exactness is however on the right:

Proposition 7.8.12. *For every M and M', we have $\pi(M \cap M') \neq \pi(M) \cap \pi(M')$ if, and only if, the sequence $0 \to \ker h_{M+M'} \to \ker h_M \oplus \ker h_{M'} \to \ker\big(k[\boldsymbol{s}] \to k[\boldsymbol{s}, \boldsymbol{r}]/(\mathcal{I}M + \mathcal{I}M')\big) \to 0$ is not exact at $\ker\big(k[\boldsymbol{s}] \to k[\boldsymbol{s}, \boldsymbol{r}]/(\mathcal{I}M + \mathcal{I}M')\big)$.*

Proof. Using Proposition 7.8.10, we get $(\mathcal{I}M + \mathcal{I}M') \cap k[\boldsymbol{s}] = (\mathcal{I}M \cap k[\boldsymbol{s}]) + (\mathcal{I}M' \cap k[\boldsymbol{s}])$ if, and only if, $\pi(M \cap M') = \pi(M) \cap \pi(M')$. □

The object encoding the potential for this loss is the cokernel.

Proposition 7.8.13. *We have* $\operatorname{coker} h_M = k[\boldsymbol{s}, \boldsymbol{r}]/(\mathcal{I}(M) + k[\boldsymbol{s}])$.

Proof. The image of $\pi\iota$ is $(\mathcal{I}(M) + k[\boldsymbol{s}])/I(M)$. The result follows by the third isomorphism theorem. □

It is important to keep in mind that all the modules in play are considered to be $k[\boldsymbol{s}]$-modules. Thus $\operatorname{coker} h_M$ is a $k[\boldsymbol{s}]$-module containing all the equations in $\mathcal{I}(M)$ with the terms containing only variables in $\boldsymbol{s}$ removed.

7.8.3 Recovering exactness

To recover exactness, we let $\rho(M)$ be $\mathcal{I}(M) + k[\boldsymbol{s}]$.

Definition 7.8.14. *We define $\rho(M)$ to be $\{r_1 p_1 + \cdots r_2 p_n : p_i \in k[\boldsymbol{s}, \boldsymbol{r}]$ and $r_1 p_1 + \cdots r_2 p_n + q \in \mathcal{I}(M)$ for some $q \in k[\boldsymbol{s}]\}$.*

We then obtain a characterization of the form:

Theorem 7.8.15. *The following sequence is exact:*

$$0 \longrightarrow \mathcal{I}\pi M + \mathcal{I}\pi M' \longrightarrow I \longrightarrow \rho(M) \cap \rho(M')/\rho(M \cup M') \longrightarrow 0$$

where $I \in k[\boldsymbol{s}]$ is such that $\mathcal{V}(I) = \pi(M \cap M')$.

Proof. Apply Proposition 7.3.14, and then Proposition 7.3.16. □

As in the case of LTI systems, the exact sequences does not necessarily split. By appropriately tensoring and localizing we can recover more refined information. This approach will not be pursued in this paper.

7.9 Concluding remarks and limitations

We conclude with three remarks. First, the development we carried in every case of systems is very preliminary. The linear case has been the most elaborated. The insight developed there also carries on, in different forms, to the other cases. Still, every case on its own is open to a lot more scrutiny. Indeed, many consequences can be further derived. Furthermore, structured instances of the problem—e.g., whenever we understand well the common part between a system and a change—can lead to a lot more intuition. Second, the systems throughout the paper live in the same universum. The theory aims for more flexibility through the notion of exactness to interconnect systems in different universa. Such a flexibility, and its benefits, are briefly discussed in the case of linear systems. The theory also extends to interconnecting multiple systems at the same time. The flexibility can be achieved by explicitly replacing inclusion relations between behaviors by morphisms between Willems systems. Finally, the cascade-like situation developed in this paper falls within a more general theory to understand cascade-like phenomena. The behavioral approach to system theory provides us with one systems theoretic interpretation of the theory. We refer the reader to the other chapters for the details.

Chapter 8

Generativity and interactional effects: the general theory

Abstract

The chapter exposes the emergence of interaction-related phenomena as a loss of exactness. It introduces the notion of generativity, and its by-product generative effects. These occur precisely when properties or features of a system behave badly under interconnection. The chapter outlines, develops and exemplifies homological methods to deal with such phenomena. The goal is to relate the behavior of the interconnected system to that of its separate components despite the presence of such phenomena.

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8.1 Introduction

Our principal concern goes into uncovering and understanding interaction-related effects—termed, *interactional effects*—that emerge when several systems interact. Our interest, in this respect, begins with understanding cascade-like effects. The intuition behind cascade-like effects may be anecdotally described as that emerging from a falling row of dominos. The fall of the first domino triggers the fall of the second domino. All dominos then come toppling down by induction. The concern has a good mathematical appeal, but it is also one that is becoming more pressing. With the recent financial crises, power blackouts and socio-political turmoil, it is becoming more essential to rigorously understand such phenomena. We however lack a solid foundation going into the heart of the matters. More importantly we do not even know what mathematical structure/pattern gives rise to the intuition present in such phenomena. It is also valid to ask what kind of mathematics and thinking is required to advance, at a fundamental level, the understanding of such problems. This chapter exposes the emergence of interaction-related phenomena as a loss of exactness. It then outlines, develops and exemplifies how to deal with such phenomena.

The chapter reintroduces the notion of *generativity*, and its byproduct *generative effects*. Cascade effects can be seen as instances of generative effects. Such effects are seen as not intrinsic to the system. They emerge from a separation between what we decide is observable from the system and what is concealed from the system. This separation is achieved by what we term a *veil*. The veil conceals parts of the system, and leaves other parts, which we term the *phenome*, observable. The phenome may be either a property, a feature or a part of the system. Generative effects are then said to occur whenever if we modify a system, or make it interact with another, we cannot explain the phenome of the newly formed system by only looking at what is observable, namely the separate phenomes of the interacting components. The mechanisms or parts concealed under the veil interact so as to produce new observables. Thus cascading phenomena arise precisely whenever property or features of systems do not behave well under interconnection.

From an engineering perspective, we want to understand the behavior of a complicated system by understanding its separate components. Our perpetual desire is that of a modular analysis: an analysis that can be separately performed on the separate components, and then combined to yield an analysis on the whole combined system. The situations of interests, exhibiting cascade-like effects, however hinder by definition such a modularity, or *compositionality.* The question we pose then is: can we recover compositionality through other means? More precisely, **how can we relate the phenome of the interconnected system to that of its separate systems, despite the presence of interaction-related phenomena?** Notions of exactness, and particularly exact sequences, will enable us to link the behavior of the big system to that of its separate components.

The notion of generative effects may be developed on two levels: a special level and a general level. It may be formalized on a special level via the use of preordered set, and particular order homomorphisms. In the case of preorders, generative effects are seen to emerge from a certain Galois connection, induced by the veil, between the space of systems and that of phenomes. Precisely, they arise whenever the veil fails to commute with joins (i.e., taking least upper bounds). The development of generative effects may also be performed on a more general level, through functorial means. The order relation is replaced by hom-sets, and the notion of minimum is replaced by that of universal arrows. In the functorial setting, generative effects emerge precisely when there is loss of exactness. The primary reason for the generality is to set-up a suitable foundation to characterize the extent of those effects, and to answer the question posed above. The extent of the loss can be measured in abelian settings via homological methods, and then used to cope with the effects. The veil is a left-exact functor encoding the phenome, and its higher derived functor encode the generativity of the system, its potential to produce interactional effects.

The chapter develops the theory of generativity at the general level. It can be seen as a more involved functorial development of the ideas presented in [Ada17a] and [Ada17b]. The chapter illustrates how functorial thinking on one end, and homological ideas on the other, allow us to achieve an understanding, pertaining to the interaction of systems, that seems elusive in engineering practice. The goal of the effort is thus dual. On the engineering end, it outlines a trail of thinking that we view essential in understanding interaction-related phenomenon. It introduces mathematical techniques (and thinking) not overtly common in engineering practice, especially those pertaining to the interaction of systems. On the mathematical end, it presents pressing engineering applications for well developed and understood theories. We hope that the ideas presented here open up a fertile ground to understand the phenomena on a more essential level.

The outline.

The chapter begins by a review of generative effects in the special level. It then generalizes to the functorial setting, and links the emergence of generative effects to a loss of exactness. Generative effects are further discussed in the case of regular categories, and then in the case of abelian categories. In the abelian case, the generativity (i.e., the systems' potential to produce effects) may then be captured by derived functors. Most formulations of generative effects are often not of an abelian nature. The approach is then to lift our formulation to an abelian setting. We define the notion of an abelian veil lift and show that every veil admits an abelian veil lift. The goal onward is to find good lifts for various classes of situations. The chapter lays out the fundamental structure into understanding and dealing with generative effects.

8.2 Review of the special level.

We review the notion of generative effects as developed through the use of preorders. The concepts in this section have been introduced in [Ada17a] and [Ada17b]. Those two references contain more examples and intuition than what we can account for in this section. We urge the reader to skim through some of their examples, if intuitive examples seem essential yet lacking in this section.

8.2.1 Review: Interconnection of systems.

The space of systems consists of a preorder $(S, \leq)$, namely a set S equipped with a binary relation $\leq$ that is reflexive and transitive. The reader may translate, if wished, the notion of preorder to a small category where every hom set $\hom(a, b)$ consists of at most one morphism. The set $\hom(a, b)$ is non-empty if, and only if, $a \leq b$.

Definition 8.2.1. *A system s is said to be a subsystem, or a part, of s' if $s \leq s'$.*

A *finite* collection of systems interacts and yields its least upper bound in S, if it exists. We will suppose that every finite collection of elements (possibly empty) admits a least upper bound, and will thus refer to our preorder as finitely cocomplete. As every pair of elements admit a least upper bound, the order relation will also be antisymmetric, and thus a partial order. Indeed, if $a \leq b$, then the least upper bound of a and b exists and is b. If $a \leq b$ and $b \leq a$, we then get $a = b$ by uniqueness of the least upper bound.

We denote by $a \vee b$ the least upper bound of a and b. The partially ordered set $(S, \leq)$ is then isomorphic to a join semilattice $(S, \vee)$. Interconnection of systems consists of taking joins in the semilattice.

Definition 8.2.2. *The finite collection of systems $s_1, \cdots, s_m$ interact, or are interconnected, to yield their least upper bound $s_1 \vee \cdots \vee s_m$.*

There exists a unique way of interconnecting two systems, and it is via the $\vee$ operation. The space of systems is then a join semilattice $(S, \vee)$ that admits finite joins. It is however important to keep in mind that the order relation is more fundamental than the join operation.

Remark. The semilattice S always admits a minimum element, as it is the least upper bound of the empty subset of S.

A systems-theoretic interpretation

We may recover a physical interpretation through the ideas of the behavioral approach to systems theory. A Willems' system is pair $(\mathbb{U}, \mathcal{B})$ of sets where the set $\mathbb{U}$, termed the *universum*, denotes the set of all possible outcomes or trajectories, and the set $\mathcal{B} \subseteq \mathbb{U}$ denotes the set of trajectories allowed by the dynamics or the restrictions of the system. Given two systems $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ with the same universum, their interconnection yields the system $(\mathbb{U}, \mathcal{B} \cap \mathcal{B}')$ whose behavior keeps only the trajectories allowed by both separate behaviors.

The set of behaviors—considered, for simplicity, to be the set of subsets of $\mathbb{U}$—may be ordered by reverse inclusion to yield a lattice. Indeed, $(\mathbb{U}, \mathcal{B})$ is a subsystem of $(\mathbb{U}, \mathcal{B}')$ if, and only if, $\mathcal{B} \supseteq \mathcal{B}'$ i.e., $(\mathbb{U}, \mathcal{B})$ is less restricted than $(\mathbb{U}, \mathcal{B}')$. Interconnection then corresponds to the join of the lattice, corresponding to set-intersection. We refer the reader to [Wil07], [PW98] and [Ada17d] for more details.

Remark. It also helps to think of $a \leq b$ as a being an approximation or a partial description of b. Descriptions can be combined to then yield more elaborate descriptions. This remark touches upon some intuition in Domain Theory.

8.2.2 Review: Veils and generative effects.

A theory of interconnection by itself cannot account for interactional effects. Such effects are seen to arise from a separation between what we consider to be observable from the system and what we consider to be concealed. This separation is achieved by covering the system with a *veil*. The veil is intended to hide features of the system at hand, and leave other parts, termed *phenome*, bare and observable. Interactional effects emerge whenever what remains visible cannot explain the visible happenings when the systems interact.

Definition 8.2.3. *A veil on* **System** *is a pair* (P, Φ) *where* P *is a join semilattice* $(P, \leq, \vee)$*, and* $\Phi : \textbf{System} \to P$ *is a map such that:*

V.1 The map Φ *is order-preserving, i.e., if* $s \leq s'$*, then* $\Phi s \leq \Phi s'$*.*

V.2 Every phenome admits a simplest system that explains it, i.e., the set $\{s : p \leq \Phi s\}$ *has a (unique) minimum element for every phenome* p*.*

If Φ is surjective, then the role of the veil can be interpreted as concealing mechanisms that are present in the system. Two systems are identified whenever they are identical outside what we wish to hide. If Φ is injective, then the role of the veil can be interpreted as forgetting characteristics of the systems. By forgetting characteristic, we are embedding our space of system into a larger space of possible systems. Conversely, any map Φ satisfying V.1 and V.2 admits an epi-mono factorization where both factors satisfy V.1 and V.2. In this sense, every veil is achieved through a combination of concealing mechanisms and forgetting characteristics of systems.

Definition 8.2.4 (Generative effects)**.** *A veil* (P, Φ) *is said to sustain generative effects if* $\Phi(s \vee s') \neq \Phi(s) \vee \Phi(s')$ *for some* s *and* s' *in* **System**.

The condition V.1 is essential to preserve the order structure in the situation. The condition V.2 is essential to connect the emergence of generative effects to closure and kernel operators. We view such phenomenon emerging fundamentally from closure and kernel operators. The condition V.2 can also be relaxed, paying a price through a gain in complexity and a loss of structure. We refer the reader to [Ada17b] for details and intuition on relaxing V.2, and on the connection of veils and generative effects to closure and kernel operators.

An instance of generative effects in the behavioral approach.

Let $(\mathbb{U}, \mathcal{B})$ be a Willems' system. We are interested in understanding the behavior of a subsystem of $(\mathbb{U}, \mathcal{B})$ as the greater system undergoes some change. Mathematically, we are given a projection, a surjective set-map $\pi : \mathbb{U} \to \mathbb{S}$. The subsystem of $(\mathbb{U}, \mathcal{B})$ is then $(\mathbb{S}, \pi\mathcal{B})$. As discussed, given two systems $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ with the same universum, their interconnection yields the system $(\mathbb{U}, \mathcal{B} \cap \mathcal{B}')$ whose behavior keeps only the trajectories allowed by both separate behaviors. A change in $(\mathbb{U}, \mathcal{B})$ is then regarded as a system $(\mathbb{U}, \mathcal{C})$, and incurring the change consists of constructing the system $(\mathbb{U}, \mathcal{B} \cap \mathcal{C})$. The projection π sets up a veil from $(2^{\mathbb{U}}, \supseteq)$ to $(2^{\mathbb{S}}, \supseteq)$. Generative effects are sustained by the veil as:

$$\pi(\mathcal{B} \cap \mathcal{C}) \neq \pi(\mathcal{B}) \cap \pi(\mathcal{C})$$

for some B and $\mathcal{C}$. Indeed, changes in the greater system outside the subsystem can indirectly affect the subsystem. We revisit this situation in Subsection 8.5.4 and characterize the effects in concern.

8.3 Functorial generative effects.

Functorial thinking leaks in once we accept two observations. First, the inequalities in the preorder can be seen as degenerate morphisms between systems. Second, minimum elements are degenerate versions of universal arrows. The functorial development in this section partially parallel that in the case of preorders.

8.3.1 Functorial interconnection of systems.

The space of systems will be a finitely cocomplete category **System**. A system is then an object in **System**. Interaction or interconnection of systems amounts to taking finite colimits.

Definition 8.3.1. *An interaction blueprint, or simply a blueprint, consists of a pair $(\mathcal{J}, B)$ where $\mathcal{J}$ is a* **finite** *category, and $B : \mathcal{J} \to$* **System** *is a functor.*

The blueprint dictates which systems will interact, and how they will interconnected. Interaction along a blueprint $(\mathcal{J}, B)$ will be given by a universal cocone. Let $\Delta : \mathbf{System} \to \mathbf{System}^{\mathcal{J}}$ be the diagonal functor. A cocone of B is a pair $\langle s, B \Rightarrow \Delta s \rangle$. The colimit of B is a universal cocone $\langle \varinjlim B, B \Rightarrow \Delta \varinjlim B \rangle$, an initial object in the category of cocones of B.

Definition 8.3.2. *The system resulting from the interaction along a blueprint $(\mathcal{J}, B)$ is the object $\varinjlim B$ of the colimit of the diagram B.*

As **System** admits finite coproducts, $\varinjlim B$ is functorially isomorphic to the (object of the) coequalizer of the pair:

$$\amalg_{u \in Arr(\mathcal{J})} B(\operatorname{dom} u) \rightrightarrows \amalg_{j \in Obj(\mathcal{J})} B(j) \tag{8.1}$$

where the arrows are induced by the morphisms:

$$B(\operatorname{dom} u) \xrightarrow{in_{B(\operatorname{dom} u)}} \amalg_{j \in Obj(\mathcal{J})} B(j)$$

$$B(\operatorname{dom} u) \xrightarrow{B(u)} B(\operatorname{codom} u) \xrightarrow{in_{B(\operatorname{codom} u)}} \amalg_{j \in Obj(\mathcal{J})} B(j).$$

See, for instance, in [AGV72] Exposé i, the proof of Proposition 2.3 and its corresponding section, for more information. We may then always think of system interaction as coequalizing a pair of maps. The change of blueprints, however, going from B to a pair of parallel maps will have some implications as we will see in later sections.

Regardless, we will often refer back to a category $\mathcal{J}$ of the form $\bullet \longleftarrow \bullet \longrightarrow \bullet$. A blueprint B is then a span in **System**, and interconnection amounts to taking pushouts. The interconnected system is then the object of the pushout. If the blueprint B consists of monos, then, intuitively, interconnection amounts to gluing two systems along a common subsystem.

Definition 8.3.3. *Let s be a system in* **System***, then:*

i. A subsystem of s is a pair $\langle s', s' \to s \rangle$ where $s' \to s$ is monic.

ii. A controlled-system from s is a pair $\langle s'', s \to s'' \rangle$ where $s \to s''$ is epic.

These notion can be further refined with additional properties, e.g. regularity conditions. Such refinements however will not be considered in this chapter.

Revisiting the behavioral-approach interpretation.

Instead of defining a system as a pair $(\mathbb{U}, \mathcal{B})$, as done in the behavioral approach, we may explicitly think of it as an injective map $\mathcal{B} \to \mathbb{U}$ (or an arbitrary map for *generalized systems*). We may then recast the behavioral approach to systems theory into an arrow category, e.g., $\mathbf{Set}^{\mathbf{2}}$. Interconnection of systems as seen through variable sharing consists of taking pullbacks—or dually pushouts, and generally colimits—in the arrow category. The physical notions of subsystem and controlled-system (dually) coincide with monos and epis, respectively, in the arrow category. We refer the reader to [Ada17d] for details on the interpretation through the lens of injective maps, and to [PW98] and [Wil07] for a further treatment of the behavioral approach.

8.3.2 Functorial veil and generative effects.

The veil is intended to conceal parts of the systems, and leave the phenome bare and observable. As phenomes are regarded as partially-observed systems, they are in themselves systems. They thus live in a finitely cocomplete category and admit a notion of interconnection through colimits.

Definition 8.3.4. *A veil is a pair (P, Φ) of a finitely cocomplete category P, and a functor $\Phi : \mathbf{System} \to P$, such that:*

V.2. For every object p in P, the comma category $(p \downarrow \Phi)$ whose objects are the morphisms $p \to \Phi s$ with s in **System** *admits an initial object.*

The universal arrows in V.2. induce a left adjoint to the functor Φ. Indeed, every veil gives rise to an adjunction between **System** and the category of phenomes P. Monads and comonads also provide a good source of veils and intuition. Comonads on **System** give rise to veils that are interpreted to conceal mechanisms in the systems. Dually, monads whose Eilenberg-Moore category corresponds to **System** give rise to veils that are interpreted to concealing characteristics in the systems. We refer the reader to [Ada17b] for a thorough discussion of these interpretations, in the case of preorders.

If (P, Φ) is a veil, and $(\mathcal{J}, B)$ be an interaction blueprint, then the funtor Φ induces a (unique) map $\varinjlim \Phi B \to \Phi \varinjlim B$ in P. Indeed, $\Phi \varinjlim B$ defines a cocone over ΦB, and $\varinjlim \Phi B$ defines the universal cocone over ΦB, i.e., the initial object in the category of cocones.

Definition 8.3.5 (Generative Effects)**.** *A veil (P, Φ) is said to sustain generative effects if, and only if, the map $\varinjlim \Phi B \to \Phi \varinjlim B$ is not an isomorphism for some blueprint $(\mathcal{J}, B)$.*

Generative effects are thus in play whenever there is a discrepancy between the phenome of the interconnected system and the interconnection of the phenomes of the subsystems.

Notation 8.3.6. *To make the category of phenomes P explicit in the chapter, as done with* **System***, we will often refer to P as* **Phenome**.

8.3.3 Generative effects in regular categories.

This discrepancy is more structured when the categories in play are more structured. This subsection provides a more refined understanding of $\varinjlim \Phi B \to \Phi \varinjlim B$ in the case of regular categories.

Recall that a kernel pair of an arrow $f : a \to b$ is a (universal) pair of parallel arrows (k_1, k_2) making the following diagram a pullback square:

$$\begin{tikzcd}
k \arrow[r, "k_1"] \arrow[d, "k_2"'] & a \arrow[d, "f"] \\
a \arrow[r, "f"] & b
\end{tikzcd}$$

A regular epimorphism is an arrow that is a coequalizer for some parallel pair of arrows. In this case, a regular epimorphism is the coequalizer of its own kernel pair.

A category is said to be regular (in the sense [Bar71] Ch. 1) whenever (i) the kernel pair of every map exists and admits a coequalizer (ii) regular epimorphisms are preserved by pullbacks. Regular categories include for instance preorders, abelian categories, categories of algebras over monads, toposes, functor categories of regular categories, and slices of regular categories. Their main feature, that we care about in this chapter, is that every arrow in a regular category admits a factorization as me, a regular epimorphism e followed by a monomorphism m.

Notation 8.3.7. *We will denote, in diagrams, a regular epimorphism by a double-headed arrow and a monomorphism by a tailed arrow. Thus a regular-epi/mono factorization me of a map, where e is a regular epimorphism followed by a monomorphism m is drawn as:*

$$\begin{tikzcd} \cdot \arrow[r, two heads, "e"] & \cdot \arrow[r, tail, "m"] & \cdot \end{tikzcd}$$

Let **Phenome** and **System** be regular categories, and let Φ : **System** $\to$ **Phenome** be a veil. As finite colimits can be expressed as coequalizers, we consider in this subsection $\mathcal{J}$ to be:

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$$

Such a consideration may come at some cost of generality, and we discuss this later in this section. A blueprint then consists of two parallel arrows having the same domain and the same codomain. Let us consider the blueprint B:

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$$

If h is its coequalizer, then B factors through the kernel pair of h, and we get:

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, dotted] & b \arrow[r, two heads, "h"] & \varinjlim B \\ & c \arrow[u, shift left] \arrow[u, shift right] & \end{tikzcd}$$

Let us denote the kernel pair of h by the blueprint, or diagram, $B_{\textit{ker-pair}}$, then:

Proposition 8.3.8. *The following diagram commutes:*

$$\begin{tikzcd} & & \varinjlim \Phi B_{\textit{ker-pair}} \arrow[dr, tail] & \\ \Phi(a) \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \Phi(b) \arrow[r, two heads] \arrow[ur, two heads] & \varinjlim \Phi B \arrow[r] \arrow[u, two heads] & \Phi(\varinjlim B) \\ & \Phi(c) \arrow[u, shift left] \arrow[u, shift right] & & \end{tikzcd}$$

In particular, the arrow $\varinjlim \Phi B \to \Phi \varinjlim B$ admits a regular-epi/mono factorization:

$$\begin{tikzcd} \varinjlim \Phi B \arrow[r, two heads] & \varinjlim \Phi B_{\textit{ker-pair}} \arrow[r, tail] & \Phi(\varinjlim B) \end{tikzcd} .$$

Proof. The proof will mostly expound the epi-mono factorization of $\varinjlim \Phi B \to \Phi \varinjlim B$. The rest of the diagram is immediate. Consider a regular-epi/mono factorization of $\Phi(b) \to \Phi(\varinjlim B)$ as:

$$\begin{tikzcd} \Phi(b) \arrow[r, two heads, "e"] & d \arrow[r, tail, "m"] & \Phi(\varinjlim B) \end{tikzcd} .$$

We claim that the regular epi e coequalizes $\Phi B_{\textit{ker-pair}}$. To show that, we only need to show that $\Phi B_{\textit{ker-pair}}$ is the kernel pair of e. As the veil Φ preserves kernel pairs (being a right adjoint), the diagram $\Phi B_{\textit{ker-pair}}$ is the kernel pair of $\Phi(b) \to \Phi(\varinjlim B)$. Commutativity (i.e., $\Phi(b) \to d$ is a cocone over $\Phi B_{\textit{ker-pair}}$) then follows because m is monic. Universality also follows directly. Indeed, if e coequalizes some parallel arrows $P_{\textit{arrow}}$, then $\Phi(b) \to \Phi(\varinjlim B)$ is a cocone over $P_{\textit{arrow}}$. The pair $P_{\textit{arrow}}$ then factors through $\Phi B_{\textit{ker-pair}}$ by universality, and the factoring arrow has to be unique.

The map $\Phi(b) \to \varinjlim \Phi B_{\textit{ker-pair}}$ is a cocone over ΦB, and thus factors through $\Phi(b) \to \varinjlim \Phi B$. The unique map $\varinjlim \Phi B \to \varinjlim \Phi B_{\textit{ker-pair}}$ has to be a regular epimorphism: otherwise the factorization of $\Phi(b) \to \varinjlim \Phi B_{\textit{ker-pair}}$ would include a non-trivial mono, contradicting the fact that it is a regular epimorphism. □

Thus the aggregate of the phenomes coming from the separate systems can be seen always to appear in *quotient form* in the phenome of the interconnected system. In particular:

Corollary 8.3.9. *If B is a kernel pair, then $\varinjlim \Phi B \to \Phi \varinjlim B$ is a monomorphism.*

Proof. If B is already a kernel pair, then $\varinjlim \Phi B \to \varinjlim \Phi B_{\textit{ker-pair}}$ is an iso. □

However, generative effects are sustained whenever the phenome of the interconnected system cannot be explained by the aggregate of the phenomes coming from the separate systems.

Corollary 8.3.10. *If B is a kernel pair, then $\varinjlim \Phi B \to \Phi \varinjlim B$ is an isomorphism if, and only if, it is a regular epimorphism.*

Proof. In a regular category, every arrow that is both a regular-epimorphism and a monomorphism is also an isomorphism. □

In some concrete categories, such as **Set** and R-**Mod**, the epimorphisms coincide with surjective maps. Generative effects are then sustained, in such categories, precisely when points in $\Phi \varinjlim B$ fail to admit preimages in $\varinjlim \Phi B$.

In the case where the categories are abelian (with adequate properties), we can quantify this failure, and use this quantification to relate the phenome of the interconnected system to that of its subsystem, despite the presence of interactional effects.

Remark. The diagram in Proposition 8.3.8 as well as the statements that follow only (directly) apply to the case where blueprints are parallel arrows. Of course, every colimit can be transformed to a coequalizer of parallel arrows. This transformation however does not always yield the same $\varinjlim \Phi B$. The main reason is that coproducts are not preserved by the veil. This transformation does not even make much sense, although legitimate, in some regular categories. As an extreme example, consider doing such transformations in preorders, seen as categories.

It is possible to achieve a similar characterization for other class of colimits, mainly pushouts. But the chapter will not pursue such a characterization because such restrictions disappears in cases of interest. Specifically, in the case of abelian categories the veils are additive and preserve biproducts.

8.3.4 Coveil and cogenerative effects.

The development may be carried out by reversing the arrows. We would then arrive at the notion of *cogenerative effects*. Let **System** be a finitely complete category. Given an interaction blueprint $(\mathcal{J}, B)$. Cointerconnection consists of taking limits $\varprojlim B$. A coveil is a functor **System** → **Phenome** that admits a right adjoint. Cogenerative effects are then sustained whenever the coveil does not commute with limits. Other dual statements can be recovered accordingly.

We can further devise a contravariant version of generative effects. This chapter will not pursue this direction.

8.3.5 Recovering the special level.

One immediately recovers the special level of the theory by directly regarding a preorder as a category. All the development instantiates without change to the case of preorders.

There is however another more interesting means to recover the special level. Lattices will often emerge by considering sub-objects or quotient-objects of specific objects in a category. Thus, if we pick a particular system, its parts form a lattice and those parts can be joined to *partially* reconstruct the whole system. The first step towards such a view considers slice categories. Recall that if $\mathcal{C}$ is a category, and c_0 is an object of C, we denote by C/c_0 the category whose objects are the arrows $c \to c_0$, and morphisms are corresponding commutative triangles.

Proposition 8.3.11. *Let* $\Phi : \mathbf{System} \to \mathbf{Phenome}$ *be a veil, and let* s *be an object of* **System**. *The veil* Φ *induces a veil:*

$$\Phi/s : \mathbf{System}/s \to \mathbf{Phenome}/\Phi s$$

Proof. Let F be the left adjoint of Φ. The left adjoint of Φ/s sends $p \to \Phi(s)$ to the composite map $F(p) \to F\Phi(s) \to s$ where $F\Phi(s) \to s$ is the counit of the adjunction $F \dashv \Phi$. □

Let $(\textbf{System}/s)_{mono}$ and $(\textbf{Phenome}/\Phi s)_{mono}$ be the respective subcategories of $\textbf{System}/s$ and $\textbf{Phenome}/\Phi s$ whose objects are the monomorphisms in **System** and **Phenome**. Then:

Proposition 8.3.12. *The restriction of* Φ/s *to* $(\textbf{System}/s)_{mono}$ *induces a functor:*

$$(\Phi/s)_{mono} : (\textbf{System}/s)_{mono} \to (\textbf{Phenome}/\Phi s)_{mono}$$

Proof. Right adjoints preserve monomorphisms. □

Let us now suppose that the category **System** is regular.

Proposition 8.3.13. *If* **System** *is regular, then the functor* $(\Phi/s)_{mono}$ *is a veil.*

Proof. Let $F/\Phi s$ be the left adjoint of Φ/s. The left adjoint of $(\Phi/s)_{mono}$ sends an arrow $c \to \Phi s$ to the monic component in the regular-epi/mono factorization of $(F/\Phi s)(c \to \Phi s)$. The details are as follows. For clarity of notation, let 1 denote Φs. We show that every triangle:

$$\begin{tikzcd}
& 1 & \\
p \arrow[ur, tail] \arrow[rr] & & \Phi x \arrow[ul, tail]
\end{tikzcd}$$

in $(\textbf{Phenome}/\Phi s)_{mono}$ factors through a universal arrow (or triangle):

$$\begin{tikzcd}
& 1 & \\
p \arrow[ur, tail] \arrow[r] & \Phi s_{universal} \arrow[u, tail] \arrow[r, dotted] & \Phi x \arrow[ul, tail]
\end{tikzcd}$$

As Φ/s admits a left adjoint, we know that every arrow $p \to \Phi x$ factors through a universal arrow $p \to \Phi s_{free}$ in the slice categories considered. Consider a regular-epi/mono factorization of the arrow $s_{free} \to s$ in **System**. We get:

$$\begin{tikzcd}
s & \\
\mathrm{im}(s_{free}) \arrow[u, tail] \arrow[dr, dotted, tail] & \\
s_{free} \arrow[u, two heads] \arrow[r, dotted] & x \arrow[uul, tail]
\end{tikzcd}$$

The dotted monic arrow arises from the commutativity of the diagram and the uniqueness of a regular-epi/mono factorization. Finally, returning to our diagram in **Phenome**/1 we get:

$$\begin{tikzcd}
& 1 & \\
p \arrow[ur, tail] \arrow[r, tail] \arrow[dr] & \Phi\operatorname{im}(s_{free}) \arrow[u, tail] \arrow[r, dotted] & \Phi(x) \arrow[ul, tail] \\
& \Phi(s_{free}) \arrow[u] \arrow[ur, dotted] &
\end{tikzcd}$$

Every arrow $p \to \Phi(x)$ then factors through $p \to \Phi\operatorname{im}(s_{free})$. □

In this case, the left adjoint of $(\Phi/s)_{mono}$ need not coincide with the restriction of the left adjoint of Φ/s.

We have recovered a lattice of subobjects from a *larger* category. The opposite direction of embedding a lattice into a *larger* category can be also be investigated. This subject will be the concern of the last section on veil-lifts.

8.4 In the Abelian case.

We suppose in this section that **System** and **Phenome** are abelian categories. In an abelian setting, interconnection of systems leads to exact sequences, and generative effects becomes synonymous to a loss of exactness once the veil is applied.

8.4.1 Interconnection as exact sequences.

In an abelian setting, we may express interconnection of systems through exact sequences. For instance, consider the commutative diagram in **System**:

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C
\end{tikzcd}$$

The map h is the coequalizer of (f, g) if, and only if, the sequence:

$$A \xrightarrow{f-g} B \xrightarrow{h} C \longrightarrow 0$$

is exact. Colimits can also be expressed as coequalizing parallel pairs of arrows, as we have seen in Equation (8.1). In turn, we can then recover an exact sequence from a colimit.

In the case of pushouts, namely where $\mathcal{J}$ is of the form $\bullet \longleftarrow \bullet \longrightarrow \bullet$, we get the following characterization:

Proposition 8.4.1. *A square diagram in* **System***,*

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "b"'] & C \arrow[d, "c'"'] \\ B \arrow[r, "b'"] & D \end{tikzcd}$$

is a pushout square if, and only if, the sequence:

$$A \xrightarrow{(b,c)} B \oplus C \xrightarrow{b'-c'} D \longrightarrow 0$$

is exact. Exhaustively, the square diagram is:

i. a commutative square if, and only if, $A \xrightarrow{(b,c)} B \xrightarrow{b'-c'} C$ *is exact.*

ii. a pullback square if, and only if, $0 \to A \xrightarrow{(b,c)} B \xrightarrow{b'-c'} C$ *is exact.*

iii. a pushout-pullback square if, and only if, $0 \to A \xrightarrow{(b,c)} B \xrightarrow{b'-c'} C \to 0$ *is exact.*

Proof. The statement can be derived using the transformation in Equation 8.1. The statement and a proof of it can also be found in [Fre64] Proposition 2.53. □

Thus, interconnection of systems will generally give us an exact sequence:

$$A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$$

In case $A \to A'$ happens to be injective, we obtain a short exact sequence. If not, we may obtain an epi-mono factorization through the image of $A \to A'$ and recover a short exact sequence. Such a factorization will have implications on the phenome as we have seen in Proposition 8.3.8. Some of those implications will be discussed in the next subsection.

8.4.2 Generative effects as a loss of exactness.

Interconnection of systems has then been reduced to taking cokernel of maps. Generative effects will be sustained whenever the cokernel is not preserved when applying the veil.

Proposition 8.4.2. *Let B be a blueprint, and let $B_{parallel}$ be the parallel pair of arrows:*

$$\bigoplus_{u \in Arr(\mathcal{J})} B(\operatorname{dom} u) \rightrightarrows \bigoplus_{j \in Obj(\mathcal{J})} B(j)$$

Then $\varinjlim \Phi B$ is functorially isomorphic to $\varinjlim \Phi B_{parallel}$.

Proof. The veil is an additive functor and thus preserves biproducts. □

Of course, the coequalizer of parallel arrows is the cokernel of their difference. As a byproduct, we can reduce generative effects to a notion on exact sequences.

Proposition 8.4.3. *A veil sustains generative effects if, and only if, for some* $A \to A \to A'' \to 0$ *exact, the sequence* $\Phi A \to \Phi A' \to \Phi A'' \to 0$ *is not exact at either* $\Phi A'$ *or* $\Phi A''$*.*

Proof. Reduce a blueprint B to a corresponding parallel arrow (f, g), and then to an arrow $f - g$. We have that $\varinjlim B$ is isomorphic to $\mathrm{coker}(f - g)$, and $\varinjlim \Phi B$ is isomorphic to $\mathrm{coker}\, \Phi(f - g)$. Generative effects are sustained if for some exact sequence $A \to A' \to A'' \to 0$, the map $\Phi(A') \to \Phi(A'')$ is not the cokernel of $\Phi(A) \to \Phi(A')$. In such a case, either the map is not epi or if it is epi it is the cokernel of some other map. The sequence $\Phi A \to \Phi A' \to \Phi A'' \to 0$ is then not exact either at $\Phi A''$ or at $\Phi A'$, respectively. The converse follows by definition. □

Note that the sequence $\Phi A \to \Phi A' \to \Phi A'' \to 0$ may not be exact at both $\Phi A'$ and $\Phi A''$. The implications of such a fact will become clearer through Proposition 8.4.7 and the discussion following it.

If **System** and **Phenome** are abelian, then the veil Φ is an additive left-exact functor. Indeed, the functor Φ admits a left adjoint. When we restrict to short exact sequence, exactness is always preserves on the right and the middle.

Proposition 8.4.4. *Let* (P, Φ) *be a veil and let* $0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$ *be a short exact sequence in* **System***, then* $0 \longrightarrow \Phi A \longrightarrow \Phi A' \longrightarrow \Phi A'' \longrightarrow 0$ *is always exact at* ΦA *and* $\Phi A'$*.*

Proof. This follows from the left-exactness of Φ. □

Generative effects are then sustained by the veil whenever exactness is not preserved on the right, namely whenever Φ fails to be right-exact.

Proposition 8.4.5. *A veil* (P, Φ) *where* P *is abelian sustains generative effects if, and only if, for some exact sequence* $0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$ *in* **System***, the sequence* $0 \longrightarrow \Phi A \longrightarrow \Phi A' \longrightarrow \Phi A'' \longrightarrow 0$ *is not exact at* $\Phi A''$*.*

Proof. Suppose Φ sustains generative effects, then there is some exact sequence $A \xrightarrow{f} A' \to A'' \to 0$ where $\Phi A \to \Phi A' \to \Phi A'' \to 0$ is not exact. Consider an epi-mono factorization of $f : A \to A'$ as:

$$A \xrightarrow{e} \mathrm{im}(f) \xrightarrow{m} A'.$$

Then $0 \to \mathrm{im}(f) \to A' \to A'' \to 0$ is exact. If $0 \to \Phi\, \mathrm{im}(f) \to \Phi A' \to \Phi A'' \to 0$ is not exact at $\Phi A''$, then we are done. Otherwise, consider the sequence:

$$0 \to \ker(e) \to A \to \mathrm{im}(f) \to 0,$$

then $0 \to \Phi \ker(e) \to \Phi A \to \Phi\, \mathrm{im}(f) \to 0$ will not be exact at $\Phi\, \mathrm{im}(f)$. The converse follows by definition. □

The loss of exactness indicates that the map $\Phi A' \to \Phi A''$ is not epi despite the fact that $A' \to A''$ is epi.

Corollary 8.4.6. *A veil (P, Φ) sustains generative effects if, and only if, Φ does not always send epimorphisms to epimorphisms.*

Proof. If Φ sends epis to epis, then short exact sequences are preserved. Conversely, let $f : B \to C$ be an epi such that Φf is not an epi, and construct the exact sequence:

$$0 \to \ker f \to B \xrightarrow{f} C \to 0$$

Applying Φ to the sequences destroys exactness on the right. □

The loss exactness is crucial for interactional effects. It should not be seen as a nuisance, but rather as precisely the reason why we arrive to such an intuition in the first place. Thus our goal is not to omit it, but rather to cope with it and understand it.

One however needs to be careful while moving from right-exact sequences to short exact sequences. The care is mostly to be taken when intuitively relating loss of exactness to generative effects. Specifically, suppose the veil is applied to some right-exact sequence, causing a loss of exactness on the right. That exactness on the right however may be preserved if we first factor the sequence into a short-exact sequence, instead of it being only right-exact. This fact will have implications on what to interpret as generative effects, and would push for more refined notions of such effects.

For instance, consider a right-exact sequence:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads, "e"] & & A' \arrow[r, two heads] & A'' \arrow[r] & 0 \\
& \operatorname{im} f \arrow[ur, tail, "m"] & & &
\end{tikzcd}$$

The care can be exemplified by considering the following diagram:

Proposition 8.4.7. *The following diagram commutes:*

$$\begin{tikzcd}
& \operatorname{im} \Phi f \arrow[dr, tail] \arrow[dd, tail] & & \operatorname{coker} \Phi m \arrow[dr, tail] & \\
\Phi A \arrow[ur, two heads] \arrow[dr, "\Phi e"] & & \Phi A' \arrow[ur, two heads] \arrow[r, two heads] & \operatorname{coker} \Phi f \arrow[u, two heads] \arrow[r] & \Phi A'' \\
& \Phi \operatorname{im} f \arrow[ur, tail, "\Phi m"] & & &
\end{tikzcd}$$

Proof. Most of the diagram can be deduced from Proposition 8.3.8, by expressing certain monic arrows h as parallel arrows $(h, 0)$. The additional component is the map $\operatorname{im} \Phi f \to \Phi \operatorname{im} f$. The map is obtained through the uniqueness of the epi-mono factorization of $\Phi(e)$. □

If $\Phi(e)$ is epic, then $\operatorname{im} \Phi f \to \Phi \operatorname{im} f$ is iso. It then follows that $\operatorname{coker} \Phi f \to \operatorname{coker} \Phi m$ is iso. Otherwise, the cokernel of Φf can in general be different then the cokernel of Φm. Although exactness is lost when applying the veil to a right-exact sequence, it may not be lost if we factor it into a short exact sequence first. Indeed, although the sequence:

$$\Phi A \to \Phi A' \to \Phi A'' \to 0 \tag{8.2}$$

may not be exact at $\Phi A'$, the sequence:

$$\Phi \operatorname{im}(A \to A') \to \Phi A' \to \Phi A'' \to 0 \tag{8.3}$$

may be exact. This always happens whenever $\Phi A' \to \Phi A''$ is epi, i.e., whenever Φ preserves $A' \to A''$ as an epi. On another end, if (8.2) is not exact at $\Phi A''$ then (8.3) will also not be exact at $\Phi A''$. And in the case where (8.2) is further exact at $\Phi A'$, all the generative effects is encoded in the loss of exactness of $\Phi A''$.

In light of this note, depending on our application, we might want to refine the notion of generative effects accordingly. For instance, if not all blueprint are considered as valid interconnection patterns, the notion of generative effects can be revisited to capture a loss with respect to only the desired one. This chapter will not develop such refinements.

8.4.3 Generativity, derived functors and universality.

Let **System** be an abelian category with enough injectives, and let $(\mathbf{Phenome}, \Phi)$ be a veil with **Phenome** abelian. As Φ is left-exact, it admits right derived functors $R^n\Phi$. Recall that given an object S in **System**, we pick an injective resolution:

$$0 \to S \to I_0 \to I_1 \to I_2 \to \cdots$$

We recover a complex:

$$0 \to \Phi(I_0) \to \Phi(I_1) \to \Phi(I_2) \to \cdots$$

and then define $R^i\Phi(S)$ to be the ith cohomology object of the complex. Note that we always have $R^0\Phi(A) = \Phi(A)$.

Proposition 8.4.8. *The right derived functors $R^*\Phi$ form a cohomological δ-functor.*

Proof. A proof (of a dual statement) may be found in [Wei95] Theorem 2.4.6. □

Given an exact sequence:

$$0 \to A \to B \to C \to 0$$

in **System**, we then recover a long exact sequence:

$$0 \to \Phi(A) \to \Phi(B) \to \Phi(C) \to R^1\Phi(A) \to R^1\Phi(B) \to R^1\Phi(C) \to R^2\Phi(A) \to \cdots$$

Thus the first derived functor encodes a systems' potential for generative effects, namely its **generativity**. This idea can be further expanded, worked on and generalized. Our goal is to present the essentials of the theory.

The derived functors encode at least the information we need to cope with generative effects. They conversely get rid of all other information we do not need to cope with generative effects. They are universal objects:

Proposition 8.4.9. *The right derived functors $R^*\Phi$ form a universal δ-functor.*

Proof. A proof (of a dual statement) may be found in [Wei95] Theorem 2.4.7. □

The first derived functor, directly responsible for measuring the generativity of the system, can be further seen to be minimal in the following sense:

Proposition 8.4.10. *Let T^0 be a δ-functor such that $T^0 \simeq \Phi$, and let $\{\alpha_n : R^n\Phi \to T^n\}$ be the unique morphism of δ-functors extending the identity functor $T^0 \simeq \Phi$. Then the morphisms of α_1 are monic.*

Proof. Let A be an object in **System**. We can pick an exact sequence:

$$0 \to A \to I \to C \to 0$$

where I is injective. The object C is then the cokernel of $A \to I$. We can then construct the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & K \arrow[d] & \\
\Phi I \arrow[r] \arrow[d, "\simeq"] & \Phi C \arrow[r, "\delta"] \arrow[d, "\simeq"] & \mathcal{R}^1\Phi A \arrow[r] \arrow[d, "\alpha"] & 0 \\
\Phi I \arrow[r] & \Phi C \arrow[r] & T^1 A &
\end{tikzcd}$$

The rows and columns are exact. The object $R^1\Phi I$ is 0 as I is injective. This explains the 0 on the right of the diagram. Let us consider the smallest abelian subcategory of **Phenome** generated by the objects and arrows of the diagram. The generated abelian subcategory is small, and admits, via the Freyd-Mitchell embedding, an exact fully-faithful embedding into the category R-**Mod** of R-modules for some ring R. Let then $a \in \mathcal{R}^1\Phi A$ be such that $\alpha a = 0$. As δ is epi, a admits a preimage c in ΦC. But c belongs to the kernel of $\Phi C \to T^1 A$, and thus to the image of $\Phi(I) \to \Phi(C)$ by exactness. By exactness of $\Phi(I) \to \Phi(C) \to R^1\Phi A \to 0$, we have that c belongs to the kernel of δ and thus $a = \delta c = 0$. The map α is then a mono, and thus K is the zero object. As the embedding is exact, we have that K is the zero object in **Phenome**. The arrow α is then monic in **Phenome**. □

Every other piece of information that is used to (directly) cope with generative effects *contains* those given by the first derived functor. In some engineering applications, the veils will be such that higher cohomology objects vanish. In two particular applications, expounded in the next section, only the first derived functors are non-trivial.

Revisiting the epi-mono factorization.

We refer back to the diagram in Proposition 8.4.7, and ask the question: when is $\operatorname{coker}\Phi f \to \operatorname{coker}\Phi m$ an iso? Building on the diagram, being iso then implies that all the generative effects are encoded in the loss of exactness at the rightmost object.

Proposition 8.4.11. *The arrow* $\operatorname{coker}\Phi f \to \operatorname{coker}\Phi m$ *is an isomorphism if, and only if, the arrow* $\operatorname{im}\Phi f \to \Phi \operatorname{im} f$ *is an isomorphism.*

Proof. We have that $\operatorname{im}\Phi f$ is the kernel of $\operatorname{coker}\Phi f$ and that $\Phi \operatorname{im} f$ is the kernel of $\operatorname{coker}\Phi m$. The statement follows by the uniqueness (up to isomorphism) of kernels and cokernels. □

Let $e : A \to \operatorname{im} f$ be the canonical epi. We then have:

Proposition 8.4.12. *The arrow* Φe *is epi if, and only if,* $\operatorname{coker}\Phi f \to \operatorname{coker}\Phi m$ *is iso.*

Proof. The statement easily follows by the uniqueness of the epi-mono factorization. □

As a sufficient condition, consider the exact sequence:

$$A \xrightarrow{f} A' \to A'' \to 0$$

and derive an exact sequence:

$$0 \to \ker f \to A \to \operatorname{im} f \to 0.$$

Applying the veil on the sequence yields the complex:

$$0 \to \Phi \ker f \to \Phi A \to \Phi \operatorname{im} f \to 0$$

The sequence is exact if $R^1\Phi(\ker f) = 0$. This occurs if $\ker f$ is a projective object. In case our category **System** is a R-**Mod** with R a principal ideal domain, then: if A is a free module, then $\operatorname{coker}\Phi f \to \operatorname{coker}\Phi m$ is iso. Indeed, we have that every submodule of a free module is free. Such a situation will occurs (implicitly) later on when we discuss Tor functors and memory in linear time-invariant systems.

Mayer-Vietoris sequence for generative effects.

Consider the pushout-pullback square:

$$\begin{tikzcd}
B \arrow[r] & D \\
A \arrow[u] \arrow[r] & C \arrow[u]
\end{tikzcd}$$

We can then construct an exact sequence:

$$0 \to A \to B \oplus C \to D \to 0$$

and recover the following long exact sequence:

$$0 \to \Phi(A) \xrightarrow{f} \Phi(B) \oplus \Phi(C) \to \Phi(D) \to R^1\Phi(A) \xrightarrow{g} R^1\Phi(B) \oplus R^1\Phi(C) \to R^1\Phi(D) \to \cdots$$

We may then deduce the short exact sequence:

$$0 \to \mathrm{coker}(f) \to \Phi(D) \to \ker(g) \to 0$$

If, for instance, **Phenome** is the category of k-vector spaces, then the sequence further splits and we can recover:

$$\Phi(D) = \mathrm{coker}(f) \oplus \ker(g),$$

thus directly relating the phenome of the interconnected system to that of its separate systems. The component $\mathrm{coker}(f)$ is explained by the phenome of the separate systems, while the component $\ker(f)$ encodes the effects produced by generativity. The higher-order objects $R^i\Phi$ can, of course, be characterized by similar means.

8.5 Sources of abelian veils.

Abelian categories and their additive veils abound. We present, in this section, some sources, examples and applications of such veils.

8.5.1 Modules, Tor and Ext.

A prime example of an abelian category is the category R-**Mod** of modules over a ring R. We assume all rings in this section to be commutative with unit. Let P and S be two rings. We let **System** and **Phenome** be S-**Mod** and P-**Mod** respectively. In general, for every P-S-bimodule A we have the adjunction:

$$\hom_{P\text{-}\mathbf{Mod}}(M \otimes_P A, N) = \hom_{S\text{-}\mathbf{Mod}}(M, \hom_S(A, N))$$

Conversely, we have:

Proposition 8.5.1 (Eilenberg-Watts)**.** *Every veil* $\Phi : S\text{-}\mathbf{Mod} \to P\text{-}\mathbf{Mod}$ *is naturally isomorphic to* $\hom_P(A, -)$ *for some* P-S*-bimodule* A.

Proof. We refer the reader to [Eil60] or [Wat60] for a proof. □

Proposition 8.5.2 (Eilenberg-Watts)**.** *Every co-veil* $\Phi : S\text{-}\mathbf{Mod} \to P\text{-}\mathbf{Mod}$ *is naturally isomorphic to* $- \otimes_S A$ *for some* P-S*-bimodule* A.

Proof. We refer the reader to [Eil60] or [Wat60] for a proof. □

Every small abelian category (via the Freyd-Mitchell embedding) admits an fully-faithful exact embedding into a category of modules for some ring R. When restricting our class of systems to a set, the abelian subcategory it generates will be small and can be embedded into an R-**Mod** category for some R. As such, veils and coveils that are *preserved* when restricted to the smaller category of systems, may be expected to behaved as either hom-ing or tensor-ing.

A further source of veils and coveils arise from a ring homomorphism:

$$f : P \longrightarrow S$$

The ring map f induces a restriction of scalar functor:

$$F : P\text{-}\mathbf{Mod} \longleftarrow S\text{-}\mathbf{Mod}$$

We then have:

Proposition 8.5.3. *The functor F admits $- \otimes_P S : P$-**Mod** $\longrightarrow S$-**Mod** as a left adjoint and $\hom_P(S, -) : P$-**Mod** $\longrightarrow S$-**Mod** as a right adjoint.*

Proof. It is enough to note that F is naturally isomorphic to $\hom_S(S, -)$ and $- \otimes_S S$ when S is viewed as a P-S-bimodule. The proof is further expounded below. □

The functor F is necessarily exact then. To clear up any confusion, with the usual adjunction described at the beginning of the section, one needs to be careful in which category tensor-ing and hom-ing are performed. First, the rings S admits a P-S-bimodule structure. It is trivially an S-module, and gains the P-module structure through f. Now, the right adjoint of $- \otimes_P S$ is $\hom_S(S, -)$ taken over S-**Mod**. Dually, the left-adjoint of $\hom_P(-, S)$ is $- \otimes_S S$ taken over S-**Mod**. Notice that $\hom_S(S, A) = A \otimes_S S = A$. And when A is regarded as a P-module, it becomes isomorphic $F(A)$.

More intricate abelian categories (or at least R-**Mod** with complicated rings R) might be needed to achieve higher expressivity in systems. Another natural source of abelian categories is that of sheaves, but development along those lines will not be pursued in this chapter. Functor categories of abelian objects are abelian, and further aid us in declaring distinguished subobjects in our systems.

The category R-**Mod** is known to have enough projectives and injectives. The derived functors of $\hom(A, -)$ and $- \otimes A$ are known as Tor and Ext, respectively. Those functors are studied extensively. Although our veils and coveils on R-**Mod** categories seem to be *generic*, the modules themselves, both in **System** and in **Phenome**, may admit various systems-theoretic interpretations. We next study one interpretation of R-mod for simple rings, and the proceed to briefly discuss functor categories.

8.5.2 Tor functors and memory in LTI systems.

Recall that a Willems' system is a pair $(\mathbb{U}, \mathcal{B})$ of sets where the set $\mathbb{U}$, termed the *universum*, denotes the set of all possible outcomes or trajectories, and the set $\mathcal{B} \subseteq \mathbb{U}$ denotes the set of trajectories allowed by the dynamics or the restrictions of the system. Given two systems $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ with the same universum, their interconnection yields the system $(\mathbb{U}, \mathcal{B} \cap \mathcal{B}')$ whose behavior keeps only the trajectories allowed by both separate behaviors.

Universa and behaviors can be equipped with various mathematical structures. Let us fix a field k. Let $k[[z^{-1}]]$ be the ring of formal power series in the variable z^{-1} with coefficients in k. We use z^{-1} instead of z as it is commonly used in signals, systems and control. An element $\sum_{i=0}^{\infty} a_i z^{-i}$ of $k[[z^{-1}]]$ is to be regarded as a discrete-time signal taking value a_i at time i. A linear time-invariant (LTI) system can be regarded as pair $(\mathbb{U}, \mathcal{B})$ where $\mathbb{U}$ is the n-dimensional free module $k[[z^{-1}]]^n$ and $\mathcal{B}$ is a submodule of $\mathbb{U}$. Indeed, the sum of two signals in $\mathcal{B}$ is again in $\mathcal{B}$, and the signal $z^{-1}s$, the time-shifted version of $s \in \mathcal{B}$, is also in $\mathcal{B}$. As $k[[z^{-1}]]$ is a principal ideal domain and $\mathbb{U}$ is a free module, then every behavior $\mathcal{B} \subseteq \mathbb{U}$ is necessarily a free submodule.

We are interested in understanding the role that memory plays when two LTI systems interact. One way to understand this role is by *destroying* a systems' capacity for memory, and derive a memoryless system from the original one. Given two systems, we then compare the memoryless system derived from the interconnection of the two, to the interconnection of the memoryless systems derived from the separate systems. If those are not isomorphic, then we can say that memory plays a role. Memory will play a role whenever *destruction* of a system's capacity for memory does not commute with interconnection. We refer the reader to [Ada17e], for the details and the intuition.

In fact, we may derive the memoryless system from $(\mathbb{U}, \mathcal{B})$ by keeping the vector space $\mathcal{B}_{mem} := \{a \in k : a + z^{-1}s \in \mathcal{B}\}$. We keep only the values taken by signals at time step 0. We define:

$$\pi : k[[z^{-1}]] \to k[[z^{-1}]]/z^{-1}k[[z^{-1}]]$$

to be the canonical projection. We then have $\mathcal{B}_{mem} := \{\pi(s) : s \in \mathcal{B}\}$. The ring map π induces a restriction of scalar functor, which admits $\Phi(-) := - \otimes_{k[[z^{-1}]]} k[[z^{-1}]]/z^{-1}k[[z^{-1}]]$ as a left adjoint. We then get:

$$\Phi(\mathbb{U}/\mathcal{B}) = k^n/\mathcal{B}_{mem}$$

The functor Φ is a coveil that forgets a system's capacity for memory. It is right exact and admits left derived functors.

Proposition 8.5.4. *The nth left derived functor of* Φ *is* $\hom(k[[z^{-1}]]/z^{-1}k[[z^{-1}]], -)$ *for* $n = 1$ *and* 0 *for* $n > 1$.

Proof. For notational convenience, let R denote $k[[z^{-1}]]$ in this proof. Let M be an R-module. Consider the projective resolution of $R/z^{-1}R$:

$$0 \to R \xrightarrow{z^{-1}} R \to R/z^{-1}R \to 0$$

Then $L_*\Phi(M)$ is the the homology of the complex:

$$0 \to R \otimes M \xrightarrow{z^{-1}} R \otimes M \to 0.$$

We then get $\Phi(M) = M/z^{-1}M = M \otimes R/z^{-1}R$, $L_1\Phi(M) = \{m : z^{-1}m = 0\} = \hom(R/z^{-1}R, M)$ and $L_n\Phi(M) = 0$ for $n > 1$. □

From a pushout-pullback square:

$$\begin{tikzcd}
\mathbb{U}/\mathcal{B} \arrow[r] & \mathbb{U}/\mathcal{B}+\mathcal{B}' \\
\mathbb{U}/\mathcal{B}\cap\mathcal{B}' \arrow[u] \arrow[r] & \mathbb{U}/\mathcal{B}' \arrow[u]
\end{tikzcd}$$

we recover an exact sequence:

$$0 \longrightarrow \mathbb{U}/\mathcal{B}\cap\mathcal{B}' \longrightarrow \mathbb{U}/\mathcal{B} \oplus \mathbb{U}/\mathcal{B}' \longrightarrow \mathbb{U}/\mathcal{B}+\mathcal{B}' \longrightarrow 0\ .$$

Applying Φ generally incurs a loss of exactness on the left, and we thus recover a six-term exact sequence:

$$0 \longrightarrow L_1\Phi(\mathbb{U}/\mathcal{B}\cap\mathcal{B}') \longrightarrow L_1\Phi(\mathbb{U}/\mathcal{B}) \oplus L_1\Phi(\mathbb{U}/\mathcal{B}') \longrightarrow L_1\Phi(\mathbb{U}/\mathcal{B}+\mathcal{B}') \longrightarrow \cdots$$

$$\cdots \longrightarrow k^n/(\mathcal{B}\cap\mathcal{B}')_{mem} \longrightarrow k^n/\mathcal{B}_{mem} \oplus k^n/\mathcal{B}'_{mem} \longrightarrow k^n/(\mathcal{B}+\mathcal{B}')_{mem} \longrightarrow 0$$

The vector space $L_1\Phi(\mathbb{U}/\mathcal{B})$ encodes the signals $s \in \mathbb{U}$ such that $s \notin \mathcal{B}$ but $z^{-1}s \in \mathcal{B}$. It is characterizing the trajectories that only appear in delayed (shifted) form in $\mathcal{B}$, thus capturing the role of memory.

8.5.3 On abelian arrow categories.

Abelian arrow categories can be very simple, but provide just enough expressivity to start understanding interesting systems-theoretic situations.

Notation 8.5.5. *We denote the category $\bullet \to \bullet$ by $\mathbf{2}$.*

Let $\mathcal{A}$ be an abelian category, then the arrow category $\mathcal{A}^{\mathbf{2}}$ is abelian. Furthermore:

Proposition 8.5.6. *Let $\mathcal{I}$ be a small category. If $\mathcal{A}$ is abelian, complete (resp. cocomplete) and has enough injectives (resp. projectives), then the functor category $\mathcal{A}^{\mathcal{I}}$ has enough injectives (resp. projectives).*

Proof. See for instance [Wei95] Example 2.3.13 (and Exercise 2.3.8) for a proof. □

Specifically, the arrow category $\mathcal{A}^{\mathbf{2}}$ has enough injectives whenever $\mathcal{A}$ is complete and has enough injectives. Every arrow in an abelian category admits a kernel and a cokernel. Abelian arrow categories then naturally provide us with veils and coveils. In particular:

Proposition 8.5.7. *If $\mathcal{A}$ is an abelian category, then:*

$$\begin{tikzcd} \mathcal{A}^{\mathbf{2}} \arrow[r, shift right, "\text{coker}"'] & \mathcal{A}^{\mathbf{2}} \arrow[l, shift right, "\text{ker}"'] \end{tikzcd}$$

the functor ker *is right adjoint to* coker*.*

Proof. As every mono is a kernel of some map, it would be enough to show that the comma category $(g \downarrow \ker)$ whose objects are diagrams $g \to m$:

$$\begin{tikzcd} \bullet \arrow[r, "\alpha"] \arrow[d, "g"] & \bullet \arrow[d, tail, "m"] \\ \bullet \arrow[r, "\beta"] & \bullet \end{tikzcd}$$

with m monic, and morphisms are the corresponding commutative *triangles*, has $g \to \operatorname{im} g$ as an initial object. Indeed, $\operatorname{im} g = \ker \operatorname{coker} g$. We show that through the following commutative diagrams, using successive epi-mono factorizations:

$$\begin{tikzcd} \bullet \arrow[rr, two heads] \arrow[dr, two heads] \arrow[dd, "g"'] & & \operatorname{im}\alpha \arrow[r, tail] & \bullet \arrow[dd, tail, "m"] \\ & \operatorname{im} g \arrow[r, two heads] \arrow[dl, tail, "\mu"] & \operatorname{im}\beta\mu \arrow[u, dotted, "\simeq"] \arrow[dr, tail] & \\ \bullet \arrow[rrr, "\beta"] & & & \bullet \end{tikzcd}$$

The composition of two epis (resp. mono) is an epi (resp. mono). Epi-mono factorizations are unique, thus inducing the isomorphism $\operatorname{im}\beta\mu \to \operatorname{im}\alpha$. We have thus created an arrow $\operatorname{im} g \to \operatorname{im}\alpha$. Every diagram $g \to m$, then factors through the diagram $g \to \operatorname{im} g$, making it an initial object in the category $(g \downarrow \ker)$. The functor coker is then a left adjoint of ker. □

Both the ker and coker functors, in the setting, are seen to take a linear map in $\mathcal{A}^{\mathbf{2}}$ and produce another linear map in $\mathcal{A}^{\mathbf{2}}$. We can also regard ker as a functor from $\mathcal{A}^{\mathbf{2}}$ to $\mathcal{A}$. We first note that in general:

Proposition 8.5.8. *For every category $\mathcal{C}$, we have an adjunction:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift right, "diag : c \mapsto id_c"'] & \mathcal{C}^{\mathbf{2}} \arrow[l, shift right, "dom"'] \end{tikzcd}$$

where dom *is right adjoint to* diag*. The functor* dom *keeps the domain of the arrow, and the functor* diag *sends c to the identity arrow on c.*

Proof. Every arrow $A \to B$ in $\mathcal{A}$ factors through $A \overset{\sim}{\to} \operatorname{dom}\operatorname{diag} A$. □

As we can compose adjunctions to get adjunctions, regarding ker as a functor from $\mathcal{A}^{\mathbf{2}}$ to $\mathcal{A}$, we get:

Corollary 8.5.9. *If $\mathcal{A}$ is an abelian category, then:*

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "- \to 0"'] & \mathcal{A}^{\mathbf{2}} \arrow[l, shift right, "\ker"'] \end{tikzcd}$$

the $\ker : \mathcal{A}^{\mathbf{2}} \to \mathcal{A}$ *functor is a right adjoint to the functor sending A in $\mathcal{A}$ to the unique arrow $A \to 0$ in $\mathcal{A}^{\mathbf{2}}$.*

Proof. The functor $\operatorname{dom} \ker$ is a right adjoint to $\operatorname{coker} \operatorname{diag}$. □

The ker functor is then a veil from $\mathcal{A}^{\mathbf{2}}$ to $\mathcal{A}$, and whenever $\mathcal{A}$ is complete and admits enough injectives it admits right derived functors:

Proposition 8.5.10. *Suppose $\mathcal{A}$ is complete and has enough injectives. The first right derived functor $R^1 \ker$ of* $\ker : \mathcal{A}^{\mathbf{2}} \to \mathcal{A}$ *is* coker*. The nth right derived functors of* ker *are trivial* (0) *for $n > 1$.*

Proof. The fact that ker and coker form a δ-functor follows from one direct application of the Snake lemma. Indeed, a short exact sequence in $\mathcal{A}^{\mathbf{2}}$ is a commutative ladder of two rows and five columns. We then need to show that ker and coker form a universal δ-functor. It would be enough to prove that the coker functor is effaceable, see e.g., [Gro57] Proposition 2.2.1. An additive functor $F : \mathcal{A} \to \mathcal{B}$ is said to be effaceable if for every object A of $\mathcal{A}$ there is a monomorphism $u : A \to I$ such that $F(u) = 0$. To this end, let $f : A \to B$ be an object of $\mathcal{A}^{\mathbf{2}}$. As $\mathcal{A}$ has enough injectives, let $i : A \to I$ and $j : B \to J$ be monomorphisms into injective objects. We then construct the diagram:

$$\begin{tikzcd} A \arrow[r, "{(i,jf)}"] \arrow[d, "f"] & I \oplus J \arrow[d, "{(0,id)}"] \\ B \arrow[r, "j"] & J \end{tikzcd}$$

The diagram is clearly a monomorphism in $\mathcal{A}^{\mathbf{2}}$, and $\operatorname{coker}(I \oplus J \to J) = 0$. □

The coker functor is dually a coveil from $\mathcal{A}^{\mathbf{2}}$ to $\mathcal{A}$. Dual statements can then be recreated accordingly.

Let us develop an interpretation and a use for this (co)veil and its derived functors. We will also return to that veil in the last section on veil-lifts.

8.5.4 Understanding the behavior of subsystems.

Let us recall the instance of (co)generative effects mentioned in Subsection 8.2.2. Let $(\mathbb{U}, \mathcal{B})$ be a Willems' system. We are interested in understanding the behavior of a subsystem of $(\mathbb{U}, \mathcal{B})$ as the greater system undergoes some change. Mathematically, we are given a projection, a surjective set-map $\pi : \mathbb{U} \to \mathbb{S}$. The subsystem of $(\mathbb{U}, \mathcal{B})$ is then $(\mathbb{S}, \pi\mathcal{B})$. Given two systems $(\mathbb{U}, \mathcal{B})$ and $(\mathbb{U}, \mathcal{B}')$ with the same universum, their interconnection yields the system $(\mathbb{U}, \mathcal{B} \cap \mathcal{B}')$ whose behavior keeps only the trajectories allowed by both separate behaviors. A change in $(\mathbb{U}, \mathcal{B})$ is then regarded as a system $(\mathbb{U}, \mathcal{C})$, and incurring the change consists of constructing the system $(\mathbb{U}, \mathcal{B} \cap \mathcal{C})$. The projection π sets up a coveil from $(2^{\mathbb{U}}, \subseteq)$ to $(2^{\mathbb{S}}, \subseteq)$. Cogenerative effects are sustained by the coveil as:

$$\pi(\mathcal{B} \cap \mathcal{C}) \neq \pi(\mathcal{B}) \cap \pi(\mathcal{C})$$

for some $\mathcal{B}$ and $\mathcal{C}$. Our goal is to characterize this inequality, and mend this loss of exactness.

We restricts ourselves to an abelian setting. Let us fix k to be a field. We suppose that our universa are vector spaces over k, and the behaviors are subspaces. As the set intersection of subspaces is a subspace, the setup described extends unchanged to the case of k-vector spaces. We define kk-**Vect** to be the category of k-vector spaces with linear maps. Let $\mathbb{S}$ be a vector space. We define **System** to be kk-**Vect**$/\mathbb{S}$, the slice category whose objects are linear maps:

$$s : V \longrightarrow \mathbb{S}$$

and morphisms are commutative triangles:

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "s"'] & & U' \arrow[dl, "s'"] \\ & \mathbb{S} & \end{tikzcd}$$

Given a Willems system $(\mathbb{U}, \mathcal{B})$ and a map (non-necessarily surjective) $\pi : \mathbb{U} \to \mathbb{S}$, we represent the system as the composition:

$$\mathcal{B} \hookrightarrow \mathbb{U} \xrightarrow{\pi} \mathbb{S}$$

Notice that the cokernel of such a map (seen as an object in kk-$\mathbf{Vect^2}$) is $\mathbb{S}/\pi(\mathcal{B})$. We have thus encoded the behavior of the subsystem in the cokernel of the map. If we embed our systems, via $\iota : kk\text{-}\mathbf{Vect}/\mathbb{S} \to kk\text{-}\mathbf{Vect^2}$, in kk-$\mathbf{Vect^2}$ we would have more generally set up a coveil:

$$\text{coker} : kk\text{-}\mathbf{Vect^2} \to kk\text{-}\mathbf{Vect}$$

A pushout-pullback square in $(kk\text{-}\mathbf{Vect} \downarrow \mathbb{S})$ such as:

$$\begin{tikzcd}
\mathcal{B} \arrow[rr] & & \mathcal{B}+\mathcal{B}' \\
& \mathbb{S} \arrow[ul, "s"] \arrow[ur, "s^{+}"'] \arrow[dl, "s^{\cap}"'] \arrow[dr, "s'"] & \\
\mathcal{B}\cap\mathcal{B}' \arrow[uu] \arrow[rr] & & \mathcal{B}' \arrow[uu]
\end{tikzcd}$$

where $\mathcal{B}$ and $\mathcal{B}'$ are subspaces of $\mathbb{U}$, yields, via by application of ι, an exact sequence in $kk\text{-}\mathbf{Vect}^{\mathbf{2}}$, namely a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{B}\cap\mathcal{B}' \arrow[r] \arrow[d] & \mathcal{B}\oplus\mathcal{B}' \arrow[r] \arrow[d] & \mathcal{B}+\mathcal{B}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{S} \arrow[r] & \mathbb{S}\oplus\mathbb{S} \arrow[r] & \mathbb{S} \arrow[r] & 0
\end{tikzcd}$$

The coveil is coker : $kk\text{-}\mathbf{Vect}^{\mathbf{2}} \to kk\text{-}\mathbf{Vect}$, and its derived functors are ker for $n = 1$ and 0 for $n > 1$. We thus get a six-term exact sequence:

$$0 \to \ker s^{\cap} \to \ker s \oplus \ker s' \to \ker s^{+} \to \mathbb{S}/\pi(\mathcal{B}\cap\mathcal{B}') \to \mathbb{S}/\pi\mathcal{B} \oplus \mathbb{S}/\pi\mathcal{B}' \to \mathbb{S}/\pi(\mathcal{B}+\mathcal{B}') \to 0$$

The derived functor ker encodes the behavior *outside* the subsystem, if that subsystem was forced to take the value of zero. It is encoding how decoupled is the rest of the system from the subsystem.

The algebraic structure of the universa and the behavior can be further complicated. We refer the reader to [Ada17f] for a thorougher development of this problem and its solution.

The category $kk\text{-}\mathbf{Vect}/\mathbb{S}$ is not abelian. We implicitly *lifted* our comma category via ι to the abelian category $kk\text{-}\mathbf{Vect}^{\mathbf{2}}$. Such lifts will be developed and discussed in the next section.

Remark. The approach expounded allows us to interconnect systems living in different universa. Interconnecting systems in different universa can be performed by first generalizing the notion of a system as discussed in section 8.3.1, and then taking pullbacks in the appropriate (arrow) category. Such a capability will however not be explicitly discussed.

8.6 Abelian lifts of non-abelian veils.

An initial formulation of a situation of generative effects may force us to define categories which are non-necessarily abelian. In such settings, homological ideas do not apply directly. Although there are various ways to recover such ideas, we discuss means to lift our situation to ones that are abelian in nature.

8.6.1 Abelian lifts.

To lift our situation into an abelian setting, and have the lift be *useful*, we need at least three components. First, the interconnection of systems (and of phenomes) should be preserved in the lifted setting. Second, we should be able to recover, in the lifted setting, the information we need. Third, we need to encode generativity, the cause of generative effects, in the lifted setting.

Definition 8.6.1. *Let $(\boldsymbol{P}, \Phi)$ be a veil on $\boldsymbol{S}$. A veil-lift of Φ is a commutative diagram of functors:*

$$\begin{tikzcd}
\boldsymbol{P}_L & \boldsymbol{S}_L \arrow[l, "\Phi_L"'] \\
\boldsymbol{P} \arrow[u, "L_p"] & \boldsymbol{S} \arrow[l, "\Phi"'] \arrow[u, "L_s"']
\end{tikzcd}$$

such that:

- *The functor Φ_L is a veil.*
- *The functors L_s and L_p are faithful, conservative and preserve finite connected colimits.*

The veil-lift is said to be **abelian** *whenever both $\boldsymbol{P}_L$ and $\boldsymbol{S}_L$ are abelian categories.*

In case a veil-lift is abelian and the category $\boldsymbol{S}_L$ admits enough injectives, the right derived functors of Φ_L then exist. Under appropriate conditions, the derived functors will be seen to encode the loss of exactness in the lifted veil Φ.

We will abuse language and refer to Φ_L as the (veil-)lift of Φ, whenever L_s and L_p are clear from the context. The functors L_s and L_p will be referred to as the liftings. Veil-lifts can of course be vertically composed, by composing the liftings. Veil-lifts can also be composed horizontally by composing the veils, whenever the appropriate liftings match.

The definition provided should be viewed as a pattern. It could be weakened or strengthened as desired to suit particular needs. We will not be concerned with providing a *right* definition, as we will only be using it tangentially in this chapter. We will nevertheless show that one can always find such a lift, for every setting. The generality of the lift we construct may however limit its uses. Some remarks:

- The functor Φ_L is considered to be a veil. We may weaken the requirement to only have it be left-exact. This weakening would be enough for our purpose to get derived functors, if the respective categories admit enough injective objects.

- The properties of a functor *being conservative* and *being faithful* are not fully independent. We refer the reader to [AGV72] Exposé i, Section 6 for a treatment of some connections. The chapter will not pursue investigations along that line. We note however that any fully-faithful functor is conservative.

- One may further need to enhance an (iso) conservative functor to be conservative on (regular) monomorphisms, or (regular) epimorphisms, etc.

- A connected colimits is only a colimit over a connected category. Connected colimits then do not include coproducts. In an abelian setting, coproducts amount to *putting* the systems together, without making them interact. Such an intuition is not present in other categories, mainly preorders. When lifting, we intentionally want to *override* the intuition of coproduct in the base category, if needed, to replace it with one that can be consistent with putting the systems together, but separately. For more information on connected (co)limits we refer the reader to [Par90].

The characteristics of the veil-lift Φ_L are such that it preserves a lifted connected colimits if, and only if, it is initially preserved by the veil. Let $(\boldsymbol{P}, \Phi)$ be a veil on $\boldsymbol{S}$, and let:

$$\begin{tikzcd}
\boldsymbol{P}_L & \boldsymbol{S}_L \arrow[l, "\Phi_L"'] \\
\boldsymbol{P} \arrow[u, "L_p"] & \boldsymbol{S} \arrow[l, "\Phi"'] \arrow[u, "L_s"']
\end{tikzcd}$$

be a veil-lift.

Proposition 8.6.2. *For every blueprint $(\mathcal{J}, B)$ where $\mathcal{J}$ is connected, the map $\varinjlim \Phi B \to \Phi \varinjlim B$ is an isomorphism if, and only if, the map $\varinjlim \Phi_L L_s B \to \Phi_L \varinjlim L_s B$ is isomorphism.*

Proof. As L_p is conservative, we have $\varinjlim \Phi B \simeq \Phi \varinjlim \mathcal{B}$ if, and only if, $L_p \varinjlim \Phi B \simeq L_p \Phi \varinjlim \mathcal{B}$. We have $L_p \varinjlim \Phi B \simeq \varinjlim L_p \Phi B$ as L_p commutes with connected limits, and thus $L_p \varinjlim \Phi B \simeq \varinjlim \Phi L_s B$ as the diagram commutes. On another end, we have $L_p \Phi \varinjlim \mathcal{B} \simeq \Phi_L L_s \varinjlim \mathcal{B}$ by commutativity of the diagram. □

A similar diagram as in Proposition 8.3.8 can be recreated in appropriate settings. However, how inexactness is preserved and encoded in the lifted-veil can be a delicate matter. For instance, if the liftings also preserve connected limits, then we can expect that the inexactness is fully encoded in the lifted-veil. In the case where the lifted category of systems has enough projective, its derived functors ought to encode the potential for the loss in the base space. On another end, if connected limits are not preserved, then the lift completes the categories of systems and/or phenome by adding new objects. We may thus obtain a finer notion of a common system. This refinement may encode part of the inexactness.

For instance, consider the following pushout-pullback square in $\boldsymbol{S}$:

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & P
\end{tikzcd},$$

where the diagram:

$$\begin{tikzcd}
\Phi C \arrow[r] \arrow[d] & \Phi A \arrow[d] \\
\Phi B \arrow[r] & \Phi P
\end{tikzcd},$$

is not a pushout square (and thus only a pullback square as Φ is a veil). The veil Φ has sustained generative effects for such a blueprint $A \leftarrow C \rightarrow B$. Let Φ_L be an abelian veil-lift with lifting L_S. If the lifting L_S does not preserve pullbacks, then the lifted square in $\boldsymbol{S}_L$ factors through another pushout-pullback square:

$$\begin{tikzcd}
L_sC \arrow[rrd, bend left] \arrow[rdd, bend right] \arrow[rd, dotted] & & \\
& K \arrow[r] \arrow[d] & L_sA \arrow[d] . \\
& L_sB \arrow[r] & L_sP
\end{tikzcd}$$

If the object of the pullback is projective, which can be the case, then the sequence:

$$0 \rightarrow \Phi_L K \rightarrow \Phi_L L_s A \oplus \Phi_L L_s B \rightarrow \Phi_L L_s P \rightarrow 0$$

will be exact. Generative effects cannot be thought to be evidently sustained by the lifted-veil in such a particular instance. For a concrete instance of such a case, the reader is refered to the end of Subsection 8.6.4. We will discuss the situation more in the following section. But first we develop a simple example to show how lifts can work.

Do we need to project down? The faithful and conservative condition means that we are not losing information when lifting. Whether we would need to project down to the base space, to interpret the situation, will depend on the particular situation itself. It is often the case that the space of phenomes is simple, and thus its lift admits a simple interpretation. For instance, the phenomes that are made of finite sets can be encoded in the dimensions of a vector space. The interpretation of the lifted phenome, and the higher (co)homology objects, then lie in the dimension and can be expected to be easily derived. There is no need then to formally project down to obtain an interpretation of the lifted phenome and higher (co)homology objects.

8.6.2 A situation of contagion.

Let S be a set $\{a, b, c\}$. Given an undirected graph over S, we are interested in whether or not there is an undirected path from a to c. Neither of the following graphs G or G' contains a path from a to c.

$$G: \qquad a — b \qquad c \qquad\qquad G': \qquad a \qquad b — c$$

But if G and G' are combined together, by taking the union of their edge set, to yield:

$$G \cup G': \qquad a — b — c$$

the edges synchronize and a path emerges. This example can be interpreted as a simplified instance of contagion. A node may be either healthy or infected. Any healthy node connected to a neighboring infected node can become infected. Once a node is infected, it remains infected forever. The existence of a path then determines whether or not a node c becomes infected if a is infected. Such models are more generally studied in [Ada17c] and [Ada17b]. Of course, the set S in this example can be of arbitrary finite cardinality.

This example is one that will be shown to exhibit generative effects. Most importantly, the *natural* formulation of the situation is by nature not an abelian formulation. We will then define an appropriate abelian veil-lift, and characterize the generativity of the situation.

Functorial formulation.

We begin by defining our categories of systems and phenomes, and the veil relating them. Our systems will be undirected graphs with two distinguished vertices. Our phenome is whether or not an undirected graph contains a path connecting the distinguished vertices.

An undirected graph (V, E) over a vertex set V with edge set E is a reflexive and symmetric relation $E \subseteq V \times V$. If $G := (V, E)$ and $G' := (V', E')$ are undirected graphs, then a graph homomorphism $h : G \to G'$ is a set map $V \to V'$ such that if $(u, v) \in E$, then $(hu, hv) \in E'$. Let **Graph** be the category of finite undirected graphs and graph homomorphisms. Let $**$ be the graph on two vertices with no edge in between.

Definition 8.6.3. *We define* **System** *to be the comma category* $(** \downarrow \mathbf{Graph})$ *whose objects are graph homomorphisms* $s : ** \to G$ *and morphisms are commutative triangles:*

$$\begin{tikzcd}
 & ** \arrow[dl, "s"'] \arrow[dr, "s'"] & \\
G \arrow[rr, "h"] & & G'
\end{tikzcd}$$

The morphism $** \to G$ declares which are the distinguished vertices in G. The distinguished vertices need not be distinct.

Definition 8.6.4. *We define* **Phenome** *to be the category* $\mathbf{2}$*, namely* $0 \to 1$ *with two objects.*

Definition 8.6.5. *We define a functor* $F : \mathbf{System} \to \mathbf{Phenome}$ *that sends* s *to* 1 *if the distinguished vertices lie in the same connected components, and to* 0 *otherwise.*

The functor F is well defined. Indeed, let $s \to s'$ is a morphism of systems. If $F(s) = 1$, then there exists a path connecting the distinguished nodes. The path is then mapped to another path (via the graph homomorphism) connecting the image of the two distinguished terminals in s'. We then get $F(s') = 1$.

Recovering the veil.

The functor F is not a veil. As we shall see, we can however recover a veil by factoring F appropriately. To this end, we define **Tran-System** to be the full subcategory of **System** whose objects are arrows $** \to G$ where the edge-set of G is additionally a transitive relation (becoming an equivalence relation). The graphs in **Tran-System** are then disjoint union of cliques. Their connected components are complete graphs. The category **Tran-System** is a coreflective subcategory of **System**, i.e. the inclusion functor i : **Tran-System** $\to$ **System** admits a right adjoint functor T : **System** $\to$ **Tran-System**.

Proposition 8.6.6. *The functor F factors through a veil:*

i. There exists a unique functor Φ making the diagram:

$$\begin{tikzcd}
\textbf{Tran-System} \arrow[dr, dotted, "\Phi"] & \\
\textbf{System} \arrow[u, "T"] \arrow[r, "F"'] & \textbf{Phenome}
\end{tikzcd}$$

commute.

ii. The functor Φ is a veil.

iii. The veil Φ sustains generative effects.

Proof. (i.) The functor Φ is the restriction of F on **Tran-System**, and indeed $F(G) = \Phi(TG)$ as T does not create new connected components. (ii.) The functor Φ admits a left adjoint sending 0 to the graph $*\ *$ and 1 to the graph $* - *$. (iii.) Recall the example at the beginning of the subsection. □

The intuition of generative effects then emerges from the veil Φ. The categories **System**, **Tran-System** and **Phenome** are however not abelian. We will define an abelian veil-lift suited for the situation.

The abelian lift.

Let free be the free functor from the category of sets to the category of abelian groups.

Notation 8.6.7. *We denote by* **Ab-grp** *and* **Set** *the category of abelian groups and sets, respectively.*

If S is a set, $\mathrm{free}\, S$ is the free abelian group generated by the elements of S. The functor free lifts to a free functor free : $\mathbf{Ab\text{-}grp^2} \to \mathbf{Set^2}$ between the arrow categories.

Given a graph $G(V, E)$, we define $I_G \subseteq \mathrm{free}(V)$ to be the subgroup $\langle i - j : (i, j) \in E \rangle$.

Definition 8.6.8. *We define L :* **System** $\to \mathbf{Ab\text{-}grp^2}$ *to be the functor that sends:*

i. *a system* $s : ** \to G(V, E)$ *to* $Ls : \mathrm{free}(**) \to \mathrm{free}(V) \to \mathrm{free}(V)/I_G$.

ii. *a morphism* $s \to s'$*:*

$$\begin{tikzcd}
 & ** \arrow[dl, "s"'] \arrow[dr, "s'"] & \\
G \arrow[rr, "h"] & & G'
\end{tikzcd}$$

to a commutative diagram:

$$\begin{tikzcd}
\mathrm{free}(**) \arrow[r, "id"] \arrow[d, "Ls"] & \mathrm{free}(**) \arrow[d, "Ls'"] \\
\mathrm{free}(V)/I_G \arrow[r, "Lh"] & \mathrm{free}(V')/I_{G'}
\end{tikzcd}$$

where Lh *is the canonical induced map.*

The map Lh is always well defined. Indeed, the map h is a graph homomorphism and thus if $i - j \in I_G$, then $hi - hj \in I'_G$. We then have $h(I_G) \subseteq I_{G'}$ and so $\mathrm{free}(V) \to \mathrm{free}(V')/I_{G'}$ factors through $\mathrm{free}(V)/I_G$.

Theorem 8.6.9. *The functor* L *induces an abelian veil-lift:*

i. *The following diagram commutes:*

$$\begin{tikzcd}
\mathbf{Ab\text{-}grp} & \mathbf{Ab\text{-}grp^2} \arrow[l, "\ker"'] & \\
\mathbf{Phenome} \arrow[u, "\mathrm{free}"] & \mathbf{Tran\text{-}System} \arrow[l, "\Phi"'] \arrow[u, dotted, "L_{Tran}"] & \mathbf{System} \arrow[l, "F"'] \arrow[ul, "L"']
\end{tikzcd}$$

ii. *The functor* L_{Tran} *is faithful, conservative and preserves connected colimits.*

iii. *The functor* ker *is additive and admits a left-adjoint.*

Proof. (i.) The functor L factors through F. (ii.) The functor L_{Tran} is the restriction of L onto **Tran-System**. The functor L is faithful, conservative and preserves connected colimits. The restriction of L is then also faithful and preserves connected colimits. It is also conservative as **Tran-System** is a full subcategory of **System**. (iii.) We have shown in Corollary 8.5.9 that the kernel functor is left-exact, and admits a left adjoint sending K in **Ab-grp** to $K \to 0$ in $\mathbf{Ab\text{-}grp^2}$. □

Proposition 8.6.10. *For every system* s*, we have:*

$$\ker(Ls) = \begin{cases} \mathbb{Z} & \textit{if a path exists.} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The statement follows by construction. □

The phenome is appropriately encoded in the lift, and the loss of exactness due to the veil is encoded in the loss of exactness due to the veil-lift. The fact that pullbacks are not always preserved by the liftings does not affect much our situation.

The veil Φ admits the category **Tran-System** of clique graphs as domain. We can however study Φ just by studying F directly. The system in **System** may be viewed as a syntactical representation of the underlying system that matters. Two different syntactical representations may yield the same system for all our purposes here. Regardless, the functor F preserves colimits, and does not cause any nuisance when it comes to understand the loss of exactness of Φ through it.

The category $\textbf{Ab-grp}^{\mathbf{2}}$ admits enough injectives, and thus ker admits right derived functors.

Proposition 8.6.11. *We have* $R^1 \ker = \operatorname{coker}$ *and* $R^n \ker = 0$ *for* $n > 1$.

Proof. Follows from Proposition 8.5.10. □

The first cohomology group can be interpreted as keeping the induced subgraph on the nodes (along with all the edges in between) not connected to any of the distinguished vertices. Explicitly, consider a system $s : ** \to G$. Its lift is the linear map $Ls : \mathrm{free}(**) \to \mathrm{free}(V)/I_G$. Let $\iota : \mathrm{free}(**) \to \mathrm{free}(V)$ be the canonical inclusion induced by s. The cokernel of Ls is:

$$\mathrm{free}(V)/\big(I_G + \iota(\mathrm{free}(**))\big)$$

Let $W \subseteq V$ be the set of vertices that are not connected via a path in G to any node in $s(**)$. We then finally get:

$$\mathrm{free}(V)/\big(I_G + \iota(\mathrm{free}(**))\big) = \mathrm{free}(W)/J$$

where $J = \langle i - j : i, j \in W \text{ and } i - j \in I_G\rangle$.

Computing the cohomology objects.

As a further development, consider the following pushout square:

$$\begin{tikzcd}
s_c \arrow[r] \arrow[d] & s \arrow[d] \\
s' \arrow[r] & s_p
\end{tikzcd}$$

The lifted square in $\boldsymbol{S}_L$ factors through a pushout-pullback square:

$$\begin{tikzcd}
Ls_c \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\
& K \arrow[r] \arrow[d] & Ls \arrow[d] & \cdot \\
& Ls' \arrow[r] & Ls_p &
\end{tikzcd}$$

Note that Ls_c and K may be isomorphic, as the lifted square may already be a pullback square. The square translates into an exact sequence:

$$0 \to K \to Ls \oplus Ls' \to Ls_p \to 0$$

For ease of notation, let us denote ker by Φ_L and $R^1 \ker$ by H^1. We then obtain a six term exact sequence:

$$0 \to \Phi_L(K) \to \Phi_L(Ls) \oplus \Phi_L(Ls') \to \Phi_L(Ls_p) \to H^1(K) \to H^1(Ls) \oplus H^1(Ls') \to H^1(Ls_p) \to 0$$

We may then deduce $\Phi_L(Ls')$ from the remaining constituents of the exact sequence.

Proposition 8.6.12. *We have:*

$$\Phi(Ls_p) = \operatorname{coker}\Big(\Phi_L(K) \to \Phi_L(Ls) \oplus \Phi_L(Ls')\Big) \oplus \ker\Big(H^1(K) \to H^1(Ls) \oplus H^1(Ls')\Big)$$

Proof. We have that $\Phi_L(Ls_p)$ is either $\mathbb{Z}$ or 0. The short exact sequence *centered at* $\Phi_L(Ls_p)$ derived from the six-term sequence in cohomology then splits. □

Let us reconsider the example in the introductory paragraph of this section. Let G and G' be the following two graphs:

$$G: \qquad a \text{ --- } b \qquad c \qquad\qquad G': \qquad a' \qquad b' \text{ --- } c'$$

They induce systems s and s', and the system s_c onto which they are *glued* corresponds to the graph:

$$G \cap G': \qquad a_c \qquad b_c \qquad c_c$$

The two distinguished vertices in $**$ are mapped to a, a', a_c and c, c', c_c, separately.

Proposition 8.6.13. *We get a six-term exact sequence:*

$$0 \to 0 \to 0 \oplus 0 \to \Phi_L(Ls_p) \to \mathbb{Z} \to 0 \oplus 0 \to H^1(Ls_p) \to 0.$$

□

Then $\Phi_L(Ls_p) = \mathbb{Z}$, and $H^1(Ls_p) = 0$. It follows that the graph $G \cup G'$ admits a path from a to c as $H^1(G \cap G')$ is $\mathbb{Z}$, namely because b_c in $G \cap G'$ is connected to neither a_c nor c_c.

8.6.3 All veils admit an abelian veil-lift.

We establish in this section that every veil admits an abelian veil-lift. To avoid foundational issues, we will require the base category of system (often denoted by $\boldsymbol{S}$) to be a small category. This should not be seen as a detrimental restriction, as from an engineering outlook, we can restrict our attention to a particular set of systems.

The generality of the construction however limits the immediate use of the constructed abelian veil-lift. In particular, the adequacy of a lift relates to how the inexactness of the veil is encoded in the veil-lift. We discuss some issues pertaining to that remark at the end of Subsection 8.6.4.

We will work in this section with coveils rather than veils. As we will be using the Yoneda embedding in its contravariant form, working with co-veils saves us from multiple dualizations to opposite category. The abelian veil-lift for veils can then be directly deduced by duality.

The implications and requirements of co-generative effects are as follows. Co-interconnections are limits, and co-veils admit right adjoint functors. In the abelian case, the term epimorphism is replace by monomorphism. Thus co-generative effects occur whenever monos are not sent to monos by the coveil. All else remains unchanged. Dually, we define:

Definition 8.6.14. *Let $(\boldsymbol{P}, \Phi)$ be a coveil on $\boldsymbol{S}$. A coveil-lift of Φ is a commutative diagram of functors:*

$$\begin{tikzcd}
\boldsymbol{S}_L \arrow[r, "\Phi_L"] & \boldsymbol{P}_L \\
\boldsymbol{S} \arrow[u, "L_s"] \arrow[r, "\Phi"] & \boldsymbol{P} \arrow[u, "L_p"]
\end{tikzcd}$$

such that:

- *The functor Φ_L is a coveil.*
- *The functors L_s and L_p are faithful, conservative and preserve finite connected limits.*

The coveil-lift is said to be **abelian** *whenever both $\boldsymbol{P}_L$ and $\boldsymbol{S}_L$ are abelian categories.*

As mentioned, to avoid foundational issues, we consider our category $\boldsymbol{S}$ of systems to be small. Such a fact restrict limits taken from diagram whose index category is $\boldsymbol{S}$ to be small limits. Such a limit is used, for example, in the proof of [AGV72] Exposé i Proposition 5.1 that we evoke in the proof to follow.

Proposition 8.6.15. *If $\boldsymbol{P}$ and $\boldsymbol{S}$ are additive categories, then the Yoneda embedding $A \mapsto \hom(-, A)$ induces an abelian coveil-lift:*

$$\begin{tikzcd}
\textbf{Ab-grp}^{\boldsymbol{S}^{op}} \arrow[r, "\hat{\Phi}"] & \textbf{Ab-grp}^{\boldsymbol{P}^{op}} \\
\boldsymbol{S} \arrow[u, "h_{\boldsymbol{S}}:s\mapsto\hom(-,s)"] \arrow[r, "\Phi"] & \boldsymbol{P} \arrow[u, "h_{\boldsymbol{P}}:p\mapsto\hom(-,p)"']
\end{tikzcd}$$

Proof. As $\boldsymbol{S}$ and $\boldsymbol{P}$ are abelian, then $h_{\boldsymbol{S}}$ and $h_{\boldsymbol{P}}$ land in the category of presheaves of abelian groups. The (enriched) Yoneda embedding preserve all limits, including connected limits. As it is fully-faithful (by the Yoneda lemma), it is also conservative. The functors $h_{\boldsymbol{S}}$ and $h_{\boldsymbol{P}}$ are then liftings.

Given $\Phi : \boldsymbol{S} \to \boldsymbol{P}$, let us define the functor:

$$\Phi^* : \mathbf{Ab\text{-}grp}^{\boldsymbol{P}^{op}} \to \mathbf{Ab\text{-}grp}^{\boldsymbol{S}^{op}}$$

sending an object F in $\mathbf{Ab\text{-}grp}^{\boldsymbol{P}^{op}}$ to $F \circ \Phi$ in $\mathbf{Ab\text{-}grp}^{\boldsymbol{S}^{op}}$. By [AGV72] Exposé i Proposition 5.1, the functor Φ^* admits a left-adjoint $\Phi_!$, i.e. a coveil. It remains to show that the diagram:

$$\begin{tikzcd}
\mathbf{Ab\text{-}grp}^{\boldsymbol{S}^{op}} \arrow[r, "\Phi_!"] & \mathbf{Ab\text{-}grp}^{\boldsymbol{P}^{op}} \\
\boldsymbol{S} \arrow[u, "h_{\boldsymbol{S}}:s\mapsto\hom(-,s)"] \arrow[r, "\Phi"] & \boldsymbol{P} \arrow[u, "h_{\boldsymbol{P}}:p\mapsto\hom(-,p)"']
\end{tikzcd}$$

commutes. To this end, for every object F in $\mathbf{Ab\text{-}grp}^{\boldsymbol{P}^{op}}$, we have natural isomorphisms $\hom(h_{\boldsymbol{S}}\Phi(s), F) = F\Phi(s)$ and $\hom(\Phi_! h_{\boldsymbol{S}}(s), F) = \hom(h_{\boldsymbol{S}}(s), \Phi^* F) = F\Phi(s)$. Indeed, the proof of a typical yoneda lemma into presheafs of sets follows through unchanged. We also refer to [Kel36] Section 2.4 for a much more general statement. As $h \mapsto \hom(h, -)$ is fully-faithful (again a (covariant) yoneda embedding) and thus conservative, the natural isomophism $\hom(h_{\boldsymbol{S}}\Phi(s), -) \xrightarrow{\sim} \hom(\Phi_! h_{\boldsymbol{S}}(s), -)$ leads to a natural isomorphism $\Phi_! h_{\boldsymbol{S}}(s) \xrightarrow{\sim} h_{\boldsymbol{P}}\Phi(s)$. The diagram thus commutes. □

Furthermore, every coveil can be lifted to an additive coveil. Such a lift can be achieved through the free additive category generated by C.

For every category C, there is a universal arrow $\mathrm{Add}_C : C \to \mathrm{Add}(C)$ from C to additive categories. For a category C, we first define the category $\mathbb{Z}(C)$ whose objects are those of C, and its hom-set $\hom_{\mathbb{Z}(C)}(a, b)$ is the free abelian group generated by the hom-set $\hom_C(a, b)$. The objects of $\mathrm{Add}(C)$ are n-tuples of objects of $\mathbb{Z}(C)$, and the arrows of $\mathrm{Add}(C)$ are matrices of arrows of $\mathbb{Z}(C)$. For more details on the construction, we refer the reader to [ML98] Ch VIII, Ex 5-6.

Proposition 8.6.16. *The functor* $\mathrm{Add}_C : C \to \mathrm{Add}(C)$ *is a faithful, conservative and preserves finite connected limits.*

Proof. Those properties follow by construction. □

Notice that if C is a small category, then $\mathrm{Add}(C)$ is a small category.

Proposition 8.6.17. *The functors* $\mathrm{Add}_C : C \to \mathrm{Add}(C)$ *induce a coveil-lift:*

$$\begin{tikzcd}
\mathrm{Add}(\boldsymbol{S}) \arrow[r, "\mathrm{Add}(\Phi)"] & \mathrm{Add}(\boldsymbol{P}) \\
\boldsymbol{S} \arrow[u, "\mathrm{Add}_{\boldsymbol{S}}"] \arrow[r, "\Phi"] & \boldsymbol{P} \arrow[u, "\mathrm{Add}_{\boldsymbol{P}}"]
\end{tikzcd}$$

Proof. The functors Add_C are universal. The functor $\mathrm{Add}_{\boldsymbol{P}}\,\Phi$ then factors through $\mathrm{Add}_{\boldsymbol{S}}$, yielding the functor $\mathrm{Add}(\Phi)$. The functor $\mathrm{Add}(\Phi)$ can be described as sending an object $\oplus_i S_i$ to $\oplus \Phi(S_i)$ and the corresponding matrices of morphisms f_{ij} to matrices of $\Phi(f_{ij})$. Let U be the right adjoint of Φ, then every arrow $\oplus_j \Phi(S_j) \to \oplus_i P_i$ in $\mathrm{Add}(\boldsymbol{P})$ factors through the (canonical) universal arrow $\oplus_i \Phi F(P_i) \to \oplus_i P_i$. The functor $\mathrm{Add}(\Phi)$ then admits a right adjoint, and is thus a coveil. □

Similarly, if F is the right adjoint of Φ, then $\mathrm{Add}_{\boldsymbol{S}} \circ F$ factors through $\mathrm{Add}_{\boldsymbol{P}}$ to yield the right-adjoint of $\mathrm{Add}(\Phi)$. The hom-set equivalence follows directly as:

$$\begin{aligned}
\hom(\mathrm{Add}(\Phi) \oplus_i S_i, \oplus_j P_j) &= \hom(\oplus_i \Phi S_i, \oplus_j P_j)\\
&= \oplus_{i,j} \hom(\Phi S_i, P_j)\\
&= \oplus_{i,j} \hom(S_i, F P_j)\\
&= \hom(\oplus_i \Phi S_i, \oplus_j F P_j) = \hom(\oplus_i S_i, \mathrm{Add}(F) \oplus_j P_j)
\end{aligned}$$

Naturality in this case would also follow from the naturality of the hom equivalence connecting Φ and F. Combining the two coveil lifts, we get:

Theorem 8.6.18. *For every veil Φ, we have an abelian coveil-lift:*

$$\begin{tikzcd}
\mathbf{Ab\text{-}grp}^{\mathrm{Add}(\boldsymbol{S})^{op}} \arrow[r, "\hat{\mathrm{Add}}(\Phi)"] & \mathbf{Ab\text{-}grp}^{\mathrm{Add}(\boldsymbol{P})^{op}} \\
\boldsymbol{S} \arrow[u, "h_{\boldsymbol{S}}\,\mathrm{Add}_{\boldsymbol{S}}"] \arrow[r, "\Phi"] & \boldsymbol{P} \arrow[u, "h_{\boldsymbol{P}}\,\mathrm{Add}_{\boldsymbol{P}}"']
\end{tikzcd}$$

Proof. Combine Propositions 8.6.15 and 8.6.17. □

Proposition 8.6.19. *The category* $\mathbf{Ab\text{-}grp}^{\mathrm{Add}(\boldsymbol{S})^{op}}$ *has enough projectives.*

Proof. Since $\boldsymbol{S}$ is small, $\mathrm{Add}(\boldsymbol{S})^{op}$ is small. As **Ab-grp** is cocomplete and has enough projectives, the statement then follows by Proposition 8.5.6. □

The coveil-lift then admits left derived functors. The generality of the coveil-lift however may limit its usefulness. We discuss some limitations in the next subsection.

We did not use, throughout this subsection, the property that Φ is a (co)veil. On one end, this tells us that every functor can be lifted to a (co)veil. Such a lift can be immediately done into the category of presheaves over sets. On another end, we can make use of the property by defining adequate Grothendieck topologies on the categories. This leads us to sheaves rather than presheaves. We might lose *having enough projectives* by doing so, but we can recover it by restricting to a small subcategory of interest. Regardless, this approach will not be further pursued in this chapter. Also, the category of sheaves of abelian groups always has enough injectives, and not projective. Another line of approach thus consists of equipping our base categories with Grothendieck topologies, and utilizing the covariant (contravariant) Yoneda embedding in the case of coveil (resp. veil). Limits (resp. colimits) of representable functors need not yield representable functors. We would then need to ensure that sheafification will further lead us to preserve the interconnection.

We know that there always exists an abelian veil-lift. The goal is to find a *good* abelian veil-lift. Those will depend on the structure of the situtation. We refer the reader to [Ada17h] for different situations and examples of abelian veil-lifts.

8.6.4 Abelian lifts for the special level.

If our base categories $\boldsymbol{S}$ and $\boldsymbol{P}$ are restricted to be preorder categories, we acquire a *tighter* abelian veil-lift. If $\mathbf{Set}^{\mathcal{C}}$ denotes the category of presheaves of sets, we define free : $\mathbf{Set}^{\mathcal{C}} \to \mathbf{Ab\text{-}grp}^{\mathcal{C}}$ to be the free functor where free $F(A) = \mathbb{Z}(F(A))$ is the free abelian group generated by the set of sections. The free functor is left-adjoint to the forgetful functor $\mathbf{Ab\text{-}grp}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{C}}$. We then have:

Theorem 8.6.20. *Let $\boldsymbol{S}$ and $\boldsymbol{P}$ are preorder. For every coveil Φ, we have an abelian coveil-lift:*

$$\begin{tikzcd}
\mathbf{Ab\text{-}grp}^{\boldsymbol{S}^{op}} \arrow[r, "\Phi_!^{\mathbf{Ab\text{-}grp}}"] & \mathbf{Ab\text{-}grp}^{\boldsymbol{P}^{op}} \\
\boldsymbol{S} \arrow[u, "\mathrm{free}\, h_{\boldsymbol{S}}"] \arrow[r, "\Phi"] & \boldsymbol{P} \arrow[u, "\mathrm{free}\, h_{\boldsymbol{P}}"']
\end{tikzcd}$$

Proof. The functors $h_{\boldsymbol{S}}$ and $h_{\boldsymbol{P}}$ are liftings as they are fully-faithful, thus conservative and preserve connected limits. The functor free is faithful and conservative. It also preserves finite connected limits defined only on the representable functors, as it can be trivially seen to preserve wide-pullbacks defined only on representable functors. Indeed, a set of sections in a representable presheaf is either a singleton or an empty set. Given $\Phi : \boldsymbol{S} \to \boldsymbol{P}$, let us define the functor:

$$\Phi^* : \mathbf{Set}^{\boldsymbol{P}^{op}} \to \mathbf{Set}^{\boldsymbol{S}^{op}}$$

sending an object F in $\mathbf{Set}^{\boldsymbol{P}^{op}}$ to $F \circ \Phi$ in $\mathbf{Set}^{\boldsymbol{S}^{op}}$. The functor Φ^* can be restricted to $\mathbf{Ab\text{-}grp}^{\boldsymbol{P}^{op}}$ to yield a functor:

$$\Phi^{*\mathbf{Ab\text{-}grp}} : \mathbf{Ab\text{-}grp}^{\boldsymbol{P}^{op}} \to \mathbf{Ab\text{-}grp}^{\boldsymbol{S}^{op}}$$

By [AGV72] Exposé i Proposition 5.1., the functor $\Phi^{*\mathbf{Ab\text{-}grp}}$ admits a left-adjoint $\Phi_!^{\mathbf{Ab\text{-}grp}}$, i.e. a coveil. The functor Φ^* also admits a left-adjoint $\Phi_!$ making the following diagram commute:

$$\begin{tikzcd}
\mathbf{Set}^{\boldsymbol{S}^{op}} \arrow[r, "\Phi_!"] & \mathbf{Set}^{\boldsymbol{P}^{op}} \\
\boldsymbol{S} \arrow[u, "h_{\boldsymbol{S}}"] \arrow[r, "\Phi"] & \boldsymbol{P} \arrow[u, "h_{\boldsymbol{P}}"']
\end{tikzcd}$$

The original diagram then commutes by using [AGV72] Exposé i Proposition 5.8.3. □

Proposition 8.6.21. *The category* $\mathbf{Ab\text{-}grp}^{\boldsymbol{S}^{op}}$ *has enough projectives.*

Proof. As $\boldsymbol{S}$ is small and **Ab-grp** is cocomplete and has enough projectives, the statement then follows by Proposition 8.5.6. □

To gain a better understanding of this abelian veil-lift, we define a functor:

$$\iota : \mathbf{Ab\text{-}grp}^{\mathcal{C}^{op}} \to \mathbf{Ab\text{-}grp}^{\mathrm{Add}(\mathcal{C})^{op}}$$

sending a presheaf F to ιF such that:

i. $(\iota F)(A \oplus B) = F(A) \oplus F(B)$ for all A and B,

ii. $(\iota F)(f \oplus g) = F(f) \oplus F(g)$ for $f, g : A \to B$ and,

iii. $(\iota F)(mf + ng) = mF(f) + nF(g)$ for integers m and n.

The functor ι extends the base category $\mathcal{C}$ of the presheaf to $\mathrm{Add}(\mathcal{C})$. By construction, the functor ι is fully-faithful, conservative and exact. We then arrive at the following commutation rules whenever $\mathcal{C}$ is a preorder:

Proposition 8.6.22. *If* $\mathcal{C}$ *is a preorder, then the following diagram:*

$$\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}^{op}} \arrow[r, "\text{free}"'] & \mathbf{Ab\text{-}grp}^{\mathcal{C}^{op}} \arrow[r, "\iota"'] & \mathbf{Ab\text{-}grp}^{\mathrm{Add}(\mathcal{C})^{op}} \\
\mathcal{C} \arrow[u, "h"'] \arrow[rr, "\mathrm{Add}"'] & & \mathit{Add}(\mathcal{C}) \arrow[u, "h"'] \arrow[ul, dotted, "L"]
\end{tikzcd}$$

commutes.

Proof. The functor free $\circ h$ factors through $\mathcal{C} \to \mathrm{Add}(\mathcal{C})$ by universality, and thus L exists and is unique. Note that $\mathbb{Z}(\hom_C(a,b)) = \hom_{\mathit{Add}(C)}(\mathrm{Add}_C\, a, \mathrm{Add}_C\, b)$. Thus free $\circ h$ sends an object C to the presheaf:

$$C \mapsto \hom(-, \mathrm{Add}(C)) \circ \mathrm{Add}(-)$$

The functor L is then the functor sending an object C_{add} in $\mathrm{Add}(\mathcal{C})$ to the presheaf with $\mathcal{C}$ as a base category:

$$C \mapsto \hom(-, C_{\mathit{add}}) \circ \mathrm{Add}(-)$$

The commutativity of the left triangle $L\iota = h$ can then be easily checked. □

Some limitations.

The generality of this abelian coveil-lift also has limitations. Some can be illustrated through the following proposition:

Proposition 8.6.23. *For every $s \in \boldsymbol{S}$, the lifted-system* $\mathrm{free} \circ h(s)$ *is projective.*

Proof. Let S denote the lifted-system $\mathrm{free} \circ h(s)$. A presheaf morphism $m : S \to G$ with domain S is uniquely determined by its component $m(s) : S(s) \to G(s)$. Thus let $S \to F$ be a morphism, and suppose $G \to F$ is an epic morphism. We have $S(s) = \mathbb{Z}$, and the map $S(s) \to F(s)$ then factors through a map $S(s) \to G(s)$ as $S(s)$ is projective. The map $S(s) \to G(s)$ induces a morphism $S \to G$, and $S \to F$ necessarily factors through $S \to G$. □

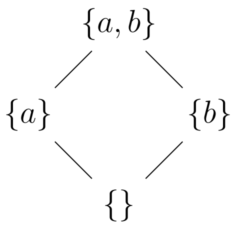
Consider then the example:

$$\Phi : 2^{\{a,b\}} \longrightarrow 2^{\{*\}}$$

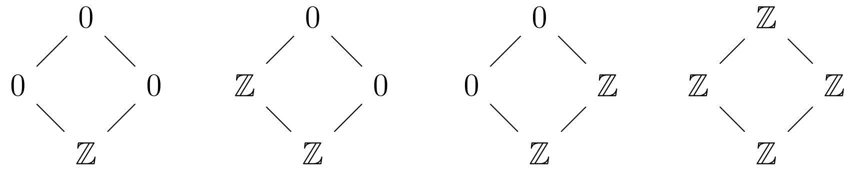
where:

$$\Phi(S) = \begin{cases} \{*\} & \text{if } S \neq \emptyset \\ \emptyset & \text{if } S = \emptyset \end{cases}$$

The map $\Phi : (2^{\{a,b\}}, \subseteq) \longrightarrow (2^{\{*\}}, \subseteq)$ is clearly a coveil. The lattice $2^{\{a,b\}}$ can be represented through a Hasse diagram as:



The map $\mathrm{free} \circ h$ sends each of the systems $\{\}$, $\{a\}$, $\{b\}$ and $\{a, b\}$ respectively to:



Cogenerative effects are sustained as:

$$\Phi(\{a\} \cap \{b\}) \neq \Phi(\{a\}) \cap \Phi(\{b\})$$

In the lifted-space, we recover an exact sequence:

$$0 \to \mathrm{free} \circ h\{\} \to \mathrm{free} \circ h\{a\} \oplus \mathrm{free} \circ h\{b\} \xrightarrow{f} \mathrm{free} \circ h\{a, b\}$$

Generative effects are sustained as the sequence:

$$0 \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$$

is not exact. But if we consider an epi-mono factorization:

$$\operatorname{free} \circ h\{a\} \oplus \operatorname{free} \circ h\{b\} \twoheadrightarrow \operatorname{im} f \rightarrowtail \operatorname{free} \circ h\{a, b\}$$

The sequence:

$$0 \to \Phi_L \operatorname{free} \circ h\{\} \to \Phi_L \operatorname{free} \circ h\{a\} \oplus \Phi_L \operatorname{free} \circ h\{b\} \to \Phi_L \operatorname{im} f \to 0$$

is exact. As the coveil-lift does not preserved colimits, it gives a refined notion of a common *subsystem*. That refinement mends some generative effects. Indeed, the lifted-system $\operatorname{im} f$ is:

$$\begin{tikzcd}
 & 0 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Z} \arrow[dr, dash] & & \mathbb{Z} \arrow[dl, dash] \\
 & \mathbb{Z} &
\end{tikzcd}$$

Applying the lifted veil Φ_L to $\operatorname{im} f$ yields $\mathbb{Z} \oplus \mathbb{Z}$.

8.7 Concluding Remarks.

Generative effects enclosing cascading phenomena is loss of exactness. A good amount of intuition and examples are required to put the notion of generative effects on solid footing. This has not been done here. We refer the reader to [Ada17a] and [Ada17b] for the details.

This chapter only presents the basics. Its ideas can be stretched, extended and modified as needed. The neverending goal is to find good abelian lifts to problem of concerns. We refer the reader to [Ada17e] and [Ada17f] for some applications, and to [Ada17h] for more information on abelian veil lifts and further applications.

Chapter 9

How to make cascade effects linear?

Abstract

It can be a common (mis)conception that cascading phenomena arise from *non-linearities*. The key message expressed throughout the thesis is that they do not. The mathematical structure underlying cascades is loss of exactness. As such it is then conceivable to lift our problematic situation to a world that is linear, keeping the effects intact. In such linear (or *abelian*) settings, tools from commutative algebra and homological algebra can be put to good use in understanding the phenomena, notably through defining (co)homology theories. We introduce the notion of an (abelian) veil-lift to encode the phenomenon (somewhat) intact in an abelian structure. We develops tools and tricks to abelianize cascading phenomena, and finally show that every situation admits an abelian veil-lift. The neverending goal is then to find tight lifts.

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9.1 Introduction

It can be a common perception that cascading phenomena arise from non-linearities, often taken to be complications of a system. We argued throughout the thesis that cascading effects, formally captured through *generative effects*, are independent of nonlinearities. They arise from a completely different mathematical structure, namely loss of exactness. It is then conceivable to find cascade-like phenomena in linear settings. Most importantly, we can linearize or abelianize the phenomena for added advantage. Homological methods and linear algebraic methods then come into the picture to characterize the phenomenon.

We refer to reader to [Ada17e] and [Ada17f] for examples on generative effects in the linear (or abelian) cases. Such examples, for instance, become immediately nonabelian problems if we remove the vector space (or R-module) consideration, and make our behaviors be only sets. Regardless, a non-abelian problem in contagion is provided in [Ada17a] and [Ada17b]. As the phenomenon is independent of whether the situation is abelian or not, it is then possible to lift the situation to one that is abelian in nature. Of course, we need the lifts to satisfy some properties. In a nutshell, we need the lift to encode and preserve the characteristics of the situation so that the lift gives us an *faithful* answer to our situation.

The topic of finding lifts is of endless bound. We develop some tricks through examples. The examples can be further developed into greater theories. This chapter can be seen to provide an array of examples and setting for lifts as discussed in [Ada17g].

9.2 Generative effects and abelian lifts.

We begin by a review of generative effects, and then introduce abelian lifts.

9.2.1 Generative effects.

Our space of systems is a join-semilattice $(\mathbf{System}, \leq, \vee, \bot)$. Two systems s and s' interact to yield $s \vee s'$. The system s is said to be a subsystem of s' if $s \leq s'$. And finally, the system $\bot$, the minimum element of the lattice, is the *identity* system $s \vee \bot = s$.

The space of systems establishes a theory of interconnection, or interaction. Yet this theory, by itself, is not enough to produce any interaction-related phenomena. Those emerge when we decide to forget *features* from the systems.

A veil is pair (P, Φ) where P is a join-semilattice, termed the space of phenomes, and $\phi : \mathbf{System} \to P$ is an order-preserving map such that $\{s : p \leq \phi(s)\}$ has a (unique) minimum element for every p. The veil is intended to conceal parts of the system and leave other parts, the phenome, bare and observable.

Definition 9.2.1. *Generative effects are said to be sustained by the veil (P, ϕ) if, and only if, $\phi(s \vee s') \neq \phi(s) \vee \phi(s')$ for some s and s'.*

Generative effects are sustained whenever the phenomes of the separate systems fail to explain the phenome of interconnected system. The features that are then concealed interact so as to produce new observables.

9.2.2 Abelian lifts.

The first step of a lift consists of transforming our systems into abelian objects, in such a way that interaction is preserved. Let $\mathcal{S}$ be a join-semilattice.

Definition 9.2.2. *A abelian lifting of $\mathcal{S}$ is a functor $L : \mathcal{S} \to \mathbf{Ab}$ into an abelian category $\mathbf{Ab}$ such that:*

i. If $s \leq s'$ and $L(s) \to L(s')$ is an isomorphism, then $s = s'$.

ii. If $f := L(s \leq s \vee s')$ and $f' := L(s' \leq s \vee s')$ then:

$$L(\bot) \longrightarrow L(s) \oplus L(s') \xrightarrow{f-f'} L(s \vee s') \longrightarrow 0$$

is an exact sequence.

The exactness condition in (ii.) is equivalent to the pair of conditions: (ii.1) the map $f - f'$ is surjective and (ii.2) every pair $(a, a') \in L(s) \oplus L(s')$ with $f(a) = f(a')$ admits a preimage in $L(\bot)$. A lifting lifts the semilattice into an abelian setting keeping the construction of systems intact. The criterion (i.) removes the case of simply using the trivial 0 abelian category of one object, and ensures that we are indeed encoding the systems without destroying information we need. A lifting by itself is not enough, more importantly, we would need to lift our veil to an abelian setting:

Definition 9.2.3. *Let $\boldsymbol{S}$ be the join semilattice of systems, and let $(\boldsymbol{P}, \Phi)$ be a veil on $\boldsymbol{S}$. An abelian veil-lift of Φ is a commutative diagram of functors:*

$$\begin{tikzcd}
\boldsymbol{P}_L & \boldsymbol{S}_L \arrow[l, "\Phi_L"'] \\
\boldsymbol{P} \arrow[u, "L_p"] & \boldsymbol{S} \arrow[l, "\Phi"] \arrow[u, "L_s"']
\end{tikzcd}$$

such that:

- *The categories $\boldsymbol{P}_L$ and $\boldsymbol{S}_L$ are abelian.*
- *The functor Φ_L is left-exact (and thus also additive).*
- *The functors L_s and L_p are liftings.*

The lifting and the veil-lift enables us to encode generative effects using abelian objects and concepts. The value of the lifting comes from the following proposition:

Theorem 9.2.4. *We have $\Phi(s \vee s') \neq \Phi(s) \vee \Phi(s')$ if, and only if, the sequence:*

$$\Phi_L L_s(\bot) \longrightarrow \Phi_L L_s(s) \oplus \Phi_L L_s(s') \xrightarrow{\Phi_L(f-f')} \Phi_L L_s(s \vee s') \longrightarrow 0$$

is not exact, at either $\Phi_L L_s(s \vee s')$ or $\Phi_L L_s(s) \oplus \Phi_L L_s(s')$.

Proof. We know that the sequence:

$$L_{\boldsymbol{S}}(\bot) \longrightarrow L_{\boldsymbol{S}}(s) \oplus L_{\boldsymbol{S}}(s') \xrightarrow{f-f'} L_{\boldsymbol{S}}(s \vee s') \longrightarrow 0$$

is always exact. We apply Φ_L to the sequence, and consider sequence:

$$\Phi_L L_{\boldsymbol{S}}(\bot) \longrightarrow \Phi_L L_{\boldsymbol{S}}(s) \oplus \Phi_L L_{\boldsymbol{S}}(s') \xrightarrow{\Phi_L f - \Phi_L f'} \Phi_L L_{\boldsymbol{S}}(s \vee s') \longrightarrow 0$$

Note that as Φ_L is additive, we always have:

$$\Phi_L L_{\boldsymbol{S}}(s) \oplus \Phi_L L_{\boldsymbol{S}}(s') = \Phi_L\big(L_{\boldsymbol{S}}(s) \oplus L_{\boldsymbol{S}}(s')\big).$$

By commutativity of the abelian veil-lift diagram, where $L_{\boldsymbol{P}}\Phi = \Phi_L L_{\boldsymbol{S}}$, we get the canonical sequence:

$$L_{\boldsymbol{P}}\bot \xrightarrow{h} L_{\boldsymbol{P}}\Phi(s) \oplus L_{\boldsymbol{P}}\Phi(s') \xrightarrow{g-g'} L_{\boldsymbol{P}}\Phi(s \vee s') \longrightarrow 0$$

Note that $\Phi(\bot) = \bot$ and that:

$$g = L_{\boldsymbol{P}}(\Phi(s) \leq \Phi(s \vee s')) = \Phi_L L_{\boldsymbol{S}}(s \leq s \vee s') = \Phi_L f. \tag{9.1}$$

A similar relation holds for g'.

If $\Phi(s \vee s') = \Phi(s) \vee \Phi(s')$, then the sequence (9.1) is exact by Property (ii.) of the lifting $L_{\boldsymbol{P}}$. Conversely, if the sequence (9.1) is exact, then $L_{\boldsymbol{P}}\Phi(s \vee s')$ is the cokernel of h. By property (ii) of the lifting, we have that the cokernel of h is $L_{\boldsymbol{P}}(\Phi(s) \vee \Phi(s'))$. Note that we always have a map:

$$L_{\boldsymbol{P}}\Phi(s \vee s') \to L_{\boldsymbol{P}}(\Phi(s) \vee \Phi(s'))$$

induced by $\Phi(s \vee s') \leq \Phi(s) \vee \Phi(s')$. As the cokernel is unique (up to isomorphism), the map $L_{\boldsymbol{P}}\Phi(s \vee s') \to L_{\boldsymbol{P}}(\Phi(s) \vee \Phi(s'))$ is an isomorphism. By Property (i.) of the lifting, we necessarily have $\Phi(s \vee s') = \Phi(s) \vee \Phi(s')$. □

Generative effects are now encoded in the loss of exactness. It can be subtle to know how it is encoded in this loss of exactness. That will depend on where exactness is lost. If the resulting sequence is exact $\Phi_L L_s(s) \oplus \Phi_L L_s(s')$ then inequality is fully encoded in the loss of surjectivity of the rightmost map. If not, then *factorizing* the sequence might fix part of the non-exactness issue. Specifically, notice that if $\Phi_L(f - f')$ is an epimorphism, then it is the cokernel of some map. We return to this issue in the last section of the chapter.

A main question is: does an abelian veil-lift exist for every veil? The answer is yes. The answer is also affirmative in a very general setting. We will discuss such a lift in the last section on semi-lattices. The generality of the lift however does have drawbacks as will be discussed. It is then a question of how well does a veil-lift use the structure of the problem: how *tight* of a lift can we find?

We will generally be discussing liftings and abelian veil-lifts, in this chapter, for various structures.

9.2.3 Why do we care about an abelian veil-lift?

The abelian lift allows us to encode the inequality due to generative effects into a loss of exactness in exact sequences. This loss of exactness can be measured via homological methods. We can then extract (co)homological objects from the systems that encode their potential to generate effects, their *generativity*. We can then use those objects to link the phenome of the interconnected system to its separate subsystems, through the use of long exact sequences, despite the presence of generative effects.

9.3 Sets as abelian groups.

As a start, we can encode the information we seek into the dimension of an abelian group, through free abelian group constructions.

Given a set Σ, we define $\mathbb{Z}^\Sigma$ to be the free abelian group generated by the basis Σ. The elements $\mathbb{Z}^\Sigma$ can be thought of as set maps $\Sigma \to \mathbb{Z}$, and addition $+$ in the group consists of pointwise addition of the corresponding maps. The free abelian group construction defines a canonical inclusion map $\iota : S \to \mathbb{Z}^S$ of sets, where an element a of Σ is identified with the functions that map a to 1 and everything else to 0. Given a subset $S \subseteq \mathbb{Z}^\Sigma$, we denote by $\langle S \rangle$ the subgroup generated by S. Thus if S is a subset of Σ, we abuse notation and denote by $\langle S \rangle$ to be the subgroup generated by $\iota(S)$.

Proposition 9.3.1. *The functor $2^\Sigma \to$ **Ab-grp** sending:*

- *A subset $S \subseteq \Sigma$ to $\mathbb{Z}^\Sigma / \langle S \rangle$.*
- *An inclusion $S \subseteq S'$ to the canonical surjection $\mathbb{Z}^\Sigma / \langle S \rangle \to \mathbb{Z}^\Sigma / \langle S' \rangle$.*

is a lifting.

Proof. As $\langle S \cup S' \rangle = \langle S \rangle + \langle S' \rangle$, the sequence:

$$\mathbb{Z}^\Sigma \to \mathbb{Z}^\Sigma / \langle S \rangle \oplus \mathbb{Z}^\Sigma / \langle S' \rangle \to \mathbb{Z}^\Sigma / \langle S \cup S' \rangle \to 0$$

is exact. The conditions (i.) and (ii.) of a lifting then directly follow. □

The lifting can be used to encode information in kernels or cokernels of certain maps. For instance, let us consider the following situation.

9.3.1 A simple instance of generativity.

Let Σ be a finite set. We define $\Phi : 2^{\Sigma} \to 2^{\{*\}}$ such that:

$$\Phi(S) = \begin{cases} \{*\} & \text{if } S = \Sigma \\ \emptyset & \text{if } S \neq \Sigma \end{cases}$$

The pair $(2^{\{*\}}, \Phi)$ is then a veil on 2^{Σ}. In words, although two subsets of Σ may not be equal to Σ, their union may be. The veil then obviously sustains generative effects as, for instance, $\Phi(S \cup S^C) \neq \Phi(S) \cup \Phi(S^C)$ for every proper subset S of Σ.

We can devise an appropriate lifting for the situation, that extends to an abelian veil-lift.

Proposition 9.3.2. *Let $s_1, \cdots, s_n$ be the basis elements of $\mathbb{Z}^{\Sigma}$. The functor $L : 2^{\Sigma} \to$* **Ab-grp²** *that sends:*

- *A subset $S \subseteq \Sigma$ to the linear map $\langle s_1 + \cdots + s_n \rangle \to \mathbb{Z}^{\Sigma}/\langle S \rangle$*
- *An inclusion $S \subseteq S'$ to the diagram:*

$$\begin{tikzcd}
\langle s_1 + \cdots + s_n \rangle \arrow[r, "id"] \arrow[d] & \langle s_1 + \cdots + s_n \rangle \arrow[d] \\
\mathbb{Z}^{\Sigma}/\langle S \rangle \arrow[r, "L_{\subseteq}"] & \mathbb{Z}^{\Sigma}/\langle S' \rangle
\end{tikzcd}$$

is a lifting.

Proof. Let us denote $s_1 + \cdots + s_n$ by $\boldsymbol{s}$. The following diagram commutes:

$$\begin{tikzcd}
\langle \boldsymbol{s} \rangle \arrow[r] \arrow[d] & \langle \boldsymbol{s} \rangle \oplus \langle \boldsymbol{s} \rangle \arrow[r] \arrow[d] & \langle \boldsymbol{s} \rangle \arrow[r] \arrow[d] & 0 \\
\mathbb{Z}^{\Sigma} \arrow[r] & \mathbb{Z}^{\Sigma}/\langle S \rangle \oplus \mathbb{Z}^{\Sigma}/\langle S' \rangle \arrow[r] & \mathbb{Z}^{\Sigma}/\langle S \cup S' \rangle \arrow[r] & 0
\end{tikzcd}.$$

The top row is exact. As $\langle S \cup S' \rangle = \langle S \rangle + \langle S' \rangle$, the bottom row is also exact. The conditions (i.) and (ii.) of a lifting then directly follow. □

Note that a sequence of linear maps corresponds to a ladder diagram. The systems are lifted to linear maps, and the phenomes then become encoded in the kernel of those maps.

Proposition 9.3.3. *The following commutative diagram:*

$$\begin{tikzcd}
\mathbf{Ab\text{-}grp} & \mathbf{Ab\text{-}grp^2} \arrow[l, "\ker"'] \\
2^{\{*\}} \arrow[u, "\text{free}"] & 2^{\Sigma} \arrow[l, "\Phi"'] \arrow[u, "L"']
\end{tikzcd}$$

is an abelian veil-lift.

Proof. The diagram is commutative as $\Phi(S) = \{*\}$ if, and only if, $\langle s_1 + \cdots + s_n \rangle \in \langle S \rangle$. The functor free where $\emptyset \mapsto 0$ and $\{*\} \mapsto \mathbb{Z}$ is trivially a lifting. The functor L is a lifting by Proposition 9.3.2. The ker functor is also known to be left-exact, see e.g. [Ada17g]. □

In this particular setting, we have:

Proposition 9.3.4. *For every s and s', the following sequence:*

$$0 \to L(S \cup S') \to L(S) \oplus L(S') \to L(S \cup S') \to 0$$

is exact.

Proof. We have $\langle S \cup S' \rangle = \langle S \rangle + \langle S' \rangle$ and $\langle S \cap S' \rangle = \langle S \rangle \cap \langle S' \rangle$. The obtained commutative ladder then has exact rows. □

Generative effects are sustained whenever the sequence:

$$0 \to \ker L(S \cap S') \to \ker L(S) \oplus \ker L(S') \to \ker L(S \cup S') \to 0$$

fails to be exact at $\ker L(S \vee S')$. As ker is left-exact, the sequence will however always be exact at $\ker L(S \wedge S')$ and $\ker L(S) \oplus \ker L(S')$. We can however recover the loss, and obtain a long exact sequence:

$$\begin{aligned}
0 \to \ker L(S \cap S') \to \ker L(S) \oplus \ker L(S') \to \ker L(S \cup S') \xrightarrow{\delta} \cdots \\
\cdots \xrightarrow{\delta} \operatorname{coker} L(S \cup S') \to \operatorname{coker} L(S) \oplus \operatorname{coker} L(S') \to \operatorname{coker} L(S \cup S') \to 0
\end{aligned}$$

The object $\ker L(S \cup S')$ can then be related to other constituents of the exact sequence. We refer the reader to [Ada17a] and [Ada17g] for the details.

9.3.2 A refinement.

As a refinement of the example, we can choose to decide the presence of a distinguished subset U of Σ. We can modify the veil Φ appropriately to get $\Phi_U : 2^{\Sigma} \to 2^{\{*\}}$ such that:

$$\Phi_U(S) = \begin{cases} \{*\} & \text{if } S \cap U = U \\ \emptyset & \text{if } S \cap U \neq U \end{cases}$$

The pair $(2^{\{*\}}, \Phi_U)$ is again a veil, and:

Proposition 9.3.5. *Let $u_1, \cdots, u_m$ be the elements of the basis of $\mathbb{Z}^\Sigma$ belonging to U. The functor L_U that sends:*

- *A subset S to a linear map $\langle u_1 + \cdots + u_m \rangle \to \mathbb{Z}^\Sigma / \langle S \rangle$*
- *A inclusion $S \subseteq S'$ to a morphism:*

$$\begin{tikzcd}
\langle u_1 + \cdots + u_m \rangle \arrow[r, "id"] \arrow[d] & \langle u_1 + \cdots + u_m \rangle \arrow[d] \\
\mathbb{Z}^\Sigma / \langle S \rangle \arrow[r, "L_\subseteq"] & \mathbb{Z}^\Sigma / \langle S' \rangle
\end{tikzcd}$$

is a lifting.

Proof. The proof is similar to that of Proposition 9.3.2. □

In this case, we again obtain an abelian veil-lift:

Proposition 9.3.6. *The following commutative diagram:*

$$\begin{tikzcd}
\textbf{Ab-grp} & 2^{\textbf{Ab-grp}} \arrow[l, "\ker"'] \\
2^{\{*\}} \arrow[u, "\text{free}"] & 2^\Sigma \arrow[l, "\Phi_U"] \arrow[u, "L_U"']
\end{tikzcd}$$

is an abelian veil-lift.

Proof. The proof is similar to that of Proposition 9.3.3 □

We can however obtain a tighter abelian veil-lift by use of the following lifting:

Proposition 9.3.7. *Let $u_1, \cdots, u_m$ be the elements of the basis of $\mathbb{Z}^\Sigma$ belonging to U. The functor L that sends:*

- *A subset S to a linear map $\langle u_1 + \cdots + u_m \rangle \to \mathbb{Z}^\Sigma / \langle S \rangle$*
- *A inclusion $S \subseteq S'$ to a morphism:*

$$\begin{tikzcd}
\langle u_1 + \cdots + u_m \rangle \arrow[r, "id"] \arrow[d] & \langle u_1 + \cdots + u_m \rangle \arrow[d] \\
\mathbb{Z}^U \oplus \mathbb{Z}^\Sigma / \langle S \rangle \arrow[r, "L_\subseteq"] & \mathbb{Z}^U \oplus \mathbb{Z}^\Sigma / \langle S' \rangle
\end{tikzcd}$$

is a lifting.

Proof. A direct sum of two exact sequences is exact. The diagram ladder we obtain commutes and has exact rows. The last needed detail is $\langle S \cup S' \rangle = \langle S \rangle + \langle S' \rangle$. □

The cokernel object is *smaller* than of the previous case, and keeps more exactly what we need for the system to recover exactness.

9.4 Lifting equivalence relations.

A relation on Σ is a subset of $\Sigma \times \Sigma$. A relation on Σ is said to be an equivalence relation if it is reflexive, symmetric and transitive.

The equivalence relations on Σ can be partially ordered by refinement to yield a lattice $\mathbf{Equ\text{-}Rel}(\Sigma)$. We have $E \leq E'$ if $a \sim b$ in E implies $a \sim b$ in E'. If E and E' are equivalence relation, then $E \vee E$ is the relation $E \cup E'$ followed by a transitive closure. It is the smallest transitive relation containing both E and E'.

If E is a relation on Σ, we denote by I_E the subgroup $\langle i - j : i \sim j \text{ in } E \rangle \subseteq \mathbb{Z}^\Sigma$. Thus, if E is an equivalence relation, we have $a \sim b$ in E if, and only if, $a - b \in I_E$

Proposition 9.4.1. *The functor* $L : \mathbf{Equ\text{-}Rel}(\Sigma) \to \mathbf{Ab\text{-}grp}$ *sending:*

- *A relation* E *to* $\mathbb{Z}^\Sigma / I_E$.
- *An inclusion* $E \leq E'$ *to the canonical surjective linear map* $\mathbb{Z}^\Sigma / I_E \to \mathbb{Z}^\Sigma / I_{E'}$.

is a lifting.

Proof. If E and E' are equivalence relations, we then have $I_{E \vee E'} = I_E + I_{E'}$. □

As done in the previous section, the lifting can be used to encode information in kernels and cokernels of certain maps.

9.4.1 A source of veils.

Consider a set map $f : A \to B$, then f induces a veil:

$$\mathbf{Equ\text{-}Rel}(A) \longleftarrow \mathbf{Equ\text{-}Rel}(B) : f^*$$

where $f^* E_B$ is the relation on A generated by $a \sim a'$ whenever $f(a) \sim f(a')$ in E_B.

Proposition 9.4.2. *The map* f^* *is a veil.*

Proof. The map f^* is clearly order-preserving. The set $\{E_B : E_A \leq f^* E_B\}$ admits a minimum (finest) equivalence relation on B generated by $f(a) \sim f(a')$ for all $a \sim a'$ in E_A. □

The map f^{-1} induces an equivalence relation on A where $a \sim a'$ if, and only if, $fa = fa'$. This equivalence relation corresponds to $f^*(E_0)$, where E_0 is the discrete (finest) equivalence relation on B.

The left adjoint of f^* (as part of a Galois connection) is:

$$f_! : \mathbf{Equ\text{-}Rel}(A) \longrightarrow \mathbf{Equ\text{-}Rel}(B)$$

where $f_!(E_A)$ is the equivalence relation on B generated by $f(a) \sim f(a')$ whenever $a \sim a'$ in E_A. In this case, we have $f_!(E \vee E') = f_!(E) \vee f_!(E')$. Meets, however, are not preserved.

9.4.2 Towards an abelian lift.

Let A and B be sets, and suppose $E \in \mathbf{Equ\text{-}Rel}(B)$. Recall that:

$$I_E = \langle i - j : i \sim j \text{ in } E \rangle.$$

A set map $f : A \to B$ induces a canonical linear map:

$$f : \mathbb{Z}^A \to \mathbb{Z}^B$$

The set map f also induces a veil f^* and we get:

$$I_{f^*E} = f^{-1} I_E \text{ for every } E \in \mathbf{Equ\text{-}Rel}(B).$$

We then obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
I_{f^*E} \arrow[r] \arrow[d] & I_E \arrow[d] \\
\mathbb{Z}^A \arrow[r] \arrow[d] & \mathbb{Z}^B \arrow[d] \\
\mathbb{Z}^A/I_{f^*E} \arrow[r] \arrow[d] & \mathbb{Z}^B/I_E \arrow[d] \\
0 & 0
\end{tikzcd}$$

In this case, consider the composite map:

$$\mathbb{Z}^A \to \mathbb{Z}^B \to \mathbb{Z}^B/I_E$$

Proposition 9.4.3. *The kernel of the composite map $\mathbb{Z}^A \to \mathbb{Z}^B/I_E$ is I_{f^*E}.*

Proof. By the commutativity of the diagram, we have $I_{f^*E} \subseteq \ker(\mathbb{Z}^A \to \mathbb{Z}^B/I_E)$. Indeed, every element of I_{f^*E} maps to 0 in $\mathbb{Z}^B/I_E$. To show the converse inequality, suppose x belongs to the kernel. Then $f(x) \in I_E$ as I_E is the kernel of $\mathbb{Z}^B \to \mathbb{Z}^B/I_E$. But $I_{f^*E} = f^{-1}I_E$, and so $x \in I_{f^*E}$. □

We then obatin a lifting:

Proposition 9.4.4. *The functor $L : \mathbf{Equ\text{-}Rel}(B) \to \mathbf{Ab\text{-}grp^2}$ sending:*

- *A relation E to $\mathbb{Z}^A \to \mathbb{Z}^B/I_E$.*

- *An refinement $E \leq E'$ to the diagram:*

$$\begin{tikzcd}
\mathbb{Z}^A \arrow[r, "id"] \arrow[d] & \mathbb{Z}^A \arrow[d] \\
\mathbb{Z}^B/I_E \arrow[r] & \mathbb{Z}^B/I_{E'}
\end{tikzcd}$$

is a lifting.

Proof. If E and E' are equivalence relations, we then have $I_{E \vee E'} = I_E + I_{E'}$. The rest is standard by now, and follows similar proofs of lifings. □

The lifting could be directly used to create a situation of generative effects have an abelian nature, or could also be made part of an abelian veil. We will discuss the abelian veil lift through an example.

9.4.3 A simple contagion application.

Let V be a set with two distinguished elements v and u. Given an undirected graph over the vertex set V, we are interested in whether or not there is a path from u to v in the graph. A typical interpetation is each node is either healthy or infected. If a node is a neighbor of an infected node, it becomes infected and remains infected forever. The question is whether or not u can become infected by v, and vice-versa.

Let G and G' be undirected graphs over the vertex set V. Although both may not admit a path connecting v and u, their combination $G \cup G' := (V, E \cup E')$ may. This situation sets up a veil as follows.

Let f be the canonical inclusion $\{u, v\} \to V$, it induces a veil:

$$\textbf{Equ-Rel}(\{u, v\}) \leftarrow \textbf{Equ-Rel}(V) : \Phi$$

The lattice **Equ-Rel**($\{u, v\}$), contains two elements, and is thus isomorphic to $2^{\{*\}}$. And we have:

$$\Phi(E) = \begin{cases} \{*\} & \text{if } u \sim v \text{ in } E \\ \emptyset & \text{otherwise} \end{cases}$$

An undirected graph G yields an equivalence relation where $i \sim j$ if, and only if, j and j are connected via a path. Our system should then be seen as an equivalence relation, and the phenome keeps whether u and v are connected via a path or not.

Corollary 9.4.5. *The functor L sending:*

- *an equivalence relation E to the composite $\mathbb{Z}^{\{u,v\}} \to \mathbb{Z}^V \to \mathbb{Z}^V/I_E$.*

- *a refinement $E \leq E'$ to a diagram:*

$$\begin{tikzcd}
\mathbb{Z}^{\{u,v\}} \arrow[r, "id"] \arrow[d] & \mathbb{Z}^{\{u,v\}} \arrow[d] \\
\mathbb{Z}^V/I_E \arrow[r, "L_{\leq}"] & \mathbb{Z}^V/I_{E'}
\end{tikzcd}$$

is a lifting.

We may further obtain an abelian lift:

Proposition 9.4.6. *The following commutative diagram:*

$$\begin{tikzcd}
\textbf{Ab-grp} & \textbf{Ab-grp}^{\mathbf{2}} \arrow[l, "\mathrm{ker}"'] \\
2^{\{*\}} \arrow[u, "free"] & \textbf{Equ-Rel}(V) \arrow[l, "\Phi"'] \arrow[u, "L"]
\end{tikzcd}$$

is an abelian veil-lift.

Proof. The diagram commutes as $\ker(\mathbb{Z}^{\{u,v\}} \to \mathbb{Z}^V/I_E$ is isomorphic to $\mathbb{Z}$ if, and only if, $u \sim v$ in E. Both functors free and L are lifting, and we know that ker is left-exact. □

The kernel of the map encodes the phenome, and the cokernel encodes the generative power of the system.

9.5 Equivalence relations with additional structure.

If $(\Sigma, \cdot)$ is a monoid, then free abelian group $\mathbb{Z}^\Sigma$ admits a ring structure with $\cdot$ as the multiplicative operator. A monoid homomorphism $f : M \to N$ then lifts to ring homorphism $f : \mathbb{Z}^M \to \mathbb{Z}^N$. A congruence relation on a monoid M is an equivalence relation on the underlying set of M that is compatible with the monoid structure. Specifically, a congruence relation is an equivalence relation such that if $i \sim i'$ and $j \sim j'$ then $i \cdot j \sim i' \cdot j'$. Equivalently:

Proposition 9.5.1. *An equivalence relation is a congruence relation if, and only if, for every k if $i \sim j$ then $i \cdot k \sim j \cdot k$.*

Proof. Let E be an equivalence relation and suppose that for every k if $i \sim j$ then $i \cdot k \sim j \cdot k$. We then have $i \cdot j \sim i' \cdot j \sim i' \cdot j'$. The converse is immediate. □

Let $(M, \cdot)$ be a monoid. The congruence relations can be ordered by refinement to yield a lattice **Cong-Rel**(M). The lattice **Cong-Rel**(M) is a sublattice of **Equ-Rel**(M). Indeed, the meet and joins coincide in both lattices.

We again define:

$$I_E = \langle i - j : i \sim j \text{ in } E \rangle \subseteq \mathbb{Z}^M$$

Note that as I_E is an ideal, then $i - j \in I_E$ implies $i \cdot k - j \cdot k \in I_E$. Similarly as in the case of equivalence relations, we obtain:

Proposition 9.5.2. *The functor $L : \mathbf{Cong\text{-}Rel}(M) \to R\text{-}\mathbf{Mod}$ sending:*

- *A relation E to $\mathbb{Z}^M / I_E$.*
- *An inclusion $E \leq E'$ to the canonical surjective linear map $\mathbb{Z}^M / I_E \to \mathbb{Z}^M / I_{E'}$.*

is a lifting.

Proof. If E and E' are congruence relations, we then have $I_{E \vee E'} = I_E + I_{E'}$. □

Similarly, consider a monoid homomorphism $f : M \to N$, then f induces a veil:

$$\mathbf{Cong\text{-}Rel}(M) \longleftarrow \mathbf{Cong\text{-}Rel}(N) : f^*$$

where $f^* E_N$ is the congruence relation on M generated by $m \sim m'$ whenever $f(m) \sim f(m')$ in E_N.

Proposition 9.5.3. *The map f^* is a veil.* □

The veil f^* also admits a left-adjoint $f_! : \mathbf{Cong\text{-}Rel}(M) \longrightarrow \mathbf{Cong\text{-}Rel}(N)$ as in the case of equivalence relations.

A similar treatment as that of the equivalence relations can be performed. The monoid structure allows us to achieve a tighter lift than what would have been possible by treating congruences as plain equivalence relations. A lifting can be achieved by sending a congruence relation E to the map:

$$\mathbb{Z}^M \to \mathbb{Z}^N \to \mathbb{Z}^N / I_E$$

When this map is viewed as a $\mathbb{Z}^M$-linear map, the kernel is $f^*(I_E)$.

Remark: The module $\mathbb{Z}^N / I_E$ admits an $\mathbb{Z}^M$-module structure through retriction of scalars via f.

9.5.1 Congruence relation on semilattices, and monotonicity.

A join semilattice $(L, \leq, \vee)$ is a monoid. Recall that a closure operator on a semilattice is a map $c : L \to L$ such that:

A.1. $a \leq c(a)$

A.2. if $a \leq b$, then $c(a) \leq c(b)$

A.3. $cc(a) = c(a)$

Let c be a closure operator, it induces an equivalence relation on L. Define a relation $\sim$ where $i \sim j$ if, and only if, $c(i) = c(j)$.

Proposition 9.5.4. *The relation $\sim$ is a congruence relation.*

Proof. Reflexivity and symmetry follow immediately. As for transitivity, if $i \sim k$ and $k \sim j$, then $c(i) = c(k)$ and $c(k) = c(j)$, and thus $i \sim j$. □

Furthermore,

Proposition 9.5.5. *Every element $c(i)$ is the maximum element of its equivalence class.*

Proof. The proof follows from A.1. and the antisymmetry of $\leq$. □

Closure operators appear in many instance and can be accordingly used. The most important fact is:

Proposition 9.5.6. *A closure operator is uniquely determined by its set of fixed-points.*

Proof. Every element i in the lattice has a (unique) least fixed-point greater than it. This least fixed-point is $c(i)$. □

We refer the reader to [Ada17c] for more details on closure operators, and their implication on cascading phenomena. In particular, let $\mathbf{Closure\text{-}op}(L)$ be the lattice of closure operators on L ordered $c \leq c'$ if, and only if, for all i, $c(i) \leq c'(i)$.

Proof. The map $\Phi : \mathbf{Closure\text{-}op}(L) \to L$ sending a closure operator to its least fixed-point is a veil. □

Proof. We refer to [Ada17b] Section 2 for a proof and more details. □

A closure operator can then be seen as a congruence relation, and thus admits a lifting as defined in the previous subsection. The ideal I_c is then defined as $\langle i - c(i) | i \in L \rangle$. Finally if $f : L \to L'$ is join semilattice homomorphism, then as we have seen:

Corollary 9.5.7. *The map $f^* : \mathbf{Cong\text{-}Rel}(L') \to \mathbf{Cong\text{-}Rel}(L)$ is a veil.* □

We can lift the situation accordingly to an abelian setting. As a quick example, we refer to [Ada17b] to an application of closure operators to contagion phenomena. The lifting then directly applies there.

9.6 Sheaves and sections.

This section will illustrate a simple use of sheaves. This section is by far not exauhaustive of the potential applications of sheaves, and only illustratively touches upon the very tip. The example is as follows.

9.6.1 The illustrative example.

Let G be a directed graph, and let H be a subgraph of the graph. Given a subgraph G' of G, we are interested in deciding whether or not it contains H as a subgraph. Although G' and G'' may not contain H separately, they may contain H once combined. All those in this section are assumed with labeling.

Formally, the situation is as follows. Let G be a directed graph, with vertex set V and arc set A. Let $\mathbf{Sub}(G)$ be the set of subgraphs of G. We can order $\mathbf{Sub}(G)$ by inclusion to obtain a lattice. Our space of systems is $\mathbf{Sub}(G)$, and we have a veil $\Phi : \mathbf{Sub}(G) \to 2^{\{*\}}$ such that:

$$\Phi(G) = \begin{cases} \{*\} & \text{if } G \text{ contains } H \\ \emptyset & \text{otherwise} \end{cases}$$

And indeed, generative effects are sustained in general.

We will devise an abelian veil-lift for the problem through the use of sheaves. The lattice $\mathbf{Sub}(G)$ is a distributive lattice. And indeed, we can declare a topology on the graph G by making its subgraphs its open sets. This topology will be the base space of the sheaf in concern.

9.6.2 Interlude on sheaves.

The theory of sheaves can be generally developed through the use of sites and Grothendieck topologies. We restrict here to sheaves defined over topological spaces. Sites and Grothendieck topologies will not be discussed in this chapter. Whenever X is a topological space, we will typically denote by $\mathcal{O}$ the lattice of its open subsets. The lattice $\mathcal{O}$ is always distributive.

Definition 9.6.1. *A presheaf F of abelian groups on a topological space $(X, \mathcal{O})$ consists of the following data:*

- *An abelian group $F(U)$ for every open set U.*
- *A linear map $F_{U \subseteq V} : F(V) \to F(U)$, termed restriction map, for every $U \subseteq V$.*

Such that $F_{U \subseteq U} = id_U$ for every U and $F_{U \subseteq W} = F_{U \subseteq V} \circ F_{V \subseteq W}$ if $U \subseteq V \subseteq W$. **Remark:** *Whenever $s \in V$ and $U \subseteq V$, for ease of notation, we denote $F_{U \subseteq V}(s)$ by $s|_U$. An element $s \in F(U)$ is termed a* section *of U.*

Thus, a presheaf of abelian groups is only a functor from $\mathcal{O}$ to $\mathbf{Ab\text{-}grp}^{op}$. This data may be further given *patching* requirements to yield the notion of a sheaf.

Definition 9.6.2. *A presheaf F is said to be a sheaf if:*

i. Let V be an open set, $\{U_i\}$ be an open cover of U and $s_i \in F(U_i)$ a section for each i. If $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i and j, then there exists an $s \in F(V)$ such that $s|_{U_i} = s_i$ for each i.

ii. Let V be an open set, $\{U_i\}$ be an open cover of V, and $s \in F(V)$. If $s|_{U_i} = 0$ for all i, then $s = 0$.

Note that *i.* and *ii.* together imply the existence (i.) and uniqueness (ii.) of a common section.

Most importantly, sheaves are related together through morphisms.

Definition 9.6.3. *Let F and G be pre-sheaves over a space $(T, \mathcal{O})$. A morphism $f : F \to G$ is a collection of maps $f_U : F(U) \to G(U)$, for every open U, such that the diagram:*

$$\begin{tikzcd}
F(U) \arrow[r, "f_U"] \arrow[d, "\subseteq"'] & G(U) \arrow[d, "\subseteq"'] \\
F(V) \arrow[r, "f_V"] & G(V)
\end{tikzcd}$$

commutes, for every $V \subseteq U$.

Every sheaf can be regarded as a presheaf by forgetting the *patching* conditions. Morphisms of sheaves are then simply morphisms of presheaves, whenever the domain and codomain sheaves are regarded as presheaves.

Definition 9.6.4. *A morphism of sheaves $F \to G$ is a morphism of presheaves.*

One class of morphism, we will be interested in, are those that are monic, and behave like injective maps. These yield the notion of a subsheaf.

Definition 9.6.5. *Let F be a sheaf. A subsheaf G of F is a pair (G, s) consisting of sheaf G with a morpshism $s : G \to F$ whose components s_U are all injective maps.*

Whenever the morphism (component of a subsheaf) is clear from the context, we forget about it, and consider only the object G as the subsheaf.

The sheaves (of abelian groups) over a topological space $(X, \mathcal{O})$ along with their morpshisms form a category $\mathbf{Ab\text{-}shv}(X)$.

The category $\mathbf{Ab\text{-}shv}(X)$ is abelian. For instance, the presheaf assigning the trivial 0 group to every open set is a sheaf. The morpshisms also admit a notion of kernel, cokernel and image. We then have a notion of exact sequences.

Definition 9.6.6. *A sequence of sheaves:*

$$\cdots \longrightarrow F_{i-1} \xrightarrow{f_{i-1}} F_i \xrightarrow{f_i} F_{i+1} \longrightarrow \cdots$$

is said to be exact at F_i whenever $\operatorname{im} f_{i-1} = \ker f_i$. A sequence is said to be exact if it is exact at every i.

Exactness for sheaves can be checked over *stalks*. But we will not need them in this chapter, and so we will not define them.

Finally, as X is the whole topological space, it is itself an open set.

Definition 9.6.7. *We define $\Gamma(X, -) : \mathbf{Ab\text{-}shv}(X) \to \mathbf{Ab\text{-}grp}$ to be the functor sending a sheaf F to the group $F(X)$.*

The functor $\Gamma(X, -)$ is termed the global section functor. We will be using this functor to encode the phenome in our abelian-veil lift. The global section functor is known to be left-exact but not right-exact. Whenever exactness fails on the right, it will be an indication of generativity.

We will lift our systems (the subgraphs in our case) to become sheaves. Interconnection of systems will then coincide will an exact sequence. Generative effects will then cause a loss of exactness when applying a suitable functor.

Back to graphs.

The sheaves on a graph G viewed as a topological space $(G, \mathbf{Sub}(G))$ admit a fairly easy description. They can be uniquely specified on the join irreducible components of the graph. The join irreducible components of the graph are the vertices and the edges. Every subgraph admits a unique decomposition as join irreducible components. By condition *i.* of a sheaf, the sections on all the subgraphs can be determined accordingly.

Given a directed graph G, we define the set maps $h : A \to V$ and $t : A \to V$ to send an arc to its head and tail, respectively.

Definition+Proposition 9.6.8. *A sheaf F on a the graph $(G, \mathbf{Sub}(G))$ is:*

- *an abelian group $F(v)$ for all $v \in V$.*
- *an abelian group $F(e)$ for all $e \in A$.*
- *a group homomorphism $F(e) \to F(he)$ for all $e \in A$.*
- *a group homomorphism $F(e) \to F(te)$ for all $e \in A$.*

A sheaf is then a presheaf defined over the sublattice of join-irreducible open sets of G. □

As corollaries of this *reduction*, we get:

Corollary 9.6.9. *If F is a subsheaf of F', then F'/F is the sheaf defined by $F'/F(v) = F'(v)/F(v)$ and $F'/F(e) = F'(e)/F(e)$ for all vertices v and arcs e.* □

Corollary 9.6.10. *A sequence $0 \to F \to F' \to F'' \to 0$ is an exact sequence of sheaves on G if, and only if, the sequence $0 \to F(v) \to F'(v) \to F''(v) \to 0$ is exact for all v, and the sequence $0 \to F(e) \to F'(e) \to F''(e) \to 0$ is exact for all e.* □

Our goal is to define a sheaf that captures the structure of the problem. For a throughout study of sheaves on graphs, we refer the reader to [Fri11] or [Fri15].

9.6.3 Detecting the full graph.

Let G be a weakly connected graph. Meaning that if we forget the directionality of the arrows, then the resulting undirected graph would consist of one connected component. In case, G is not weakly connected, we can simply restrict the analysis on a weakly connected component.

Given a directed graph G, we define the set maps $h : A \to V$ and $t : A \to V$ to send an arc to its head and tail, respectively.

Definition 9.6.11. *We define $\mathbb{Z}(G)$ to be the sheaf on $(G, \mathbf{Sub}(G))$ such that:*

- *$F(v) = \mathbb{Z}$ for all $v \in V$.*
- *$F(e) = \mathbb{Z}$ for all $e \in A$.*
- *$F(e) \to F(he) = id$ for all $e \in A$.*
- *$F(e) \to F(te) = id$ for all $e \in A$.*

The global section of the sheaf, for a weakly connected graph, is simply $\mathbb{Z}$.

Proposition 9.6.12. *For every weakly connected digram G, we have $\Gamma(G, \mathbb{Z}(G)) = \mathbb{Z}$.*

Proof. We have that the arcs $\{e_i\}_i$ forms an open cover of G. Consider a section s_i for each e_i. Either $s_i = s_j$ for all i, j or (as G is weakly connected) $s_i \neq s_j$ for some e_i and e_j sharing a vertex v. If $s_i = s_j$ for all i, j, then $s = s_i \in \Gamma(G, \mathbb{Z}(G))$. If $s_i \neq s_j$ for some e_i and e_j sharing a vertex v, then $s_i|_v \neq s_j|_v$ and a global section with s_i's as restrictions cannot then be constructed. □

For every graph G, non-necessarily weakly connected, the global section of $\mathbb{Z}(G)$ would yield $\mathbb{Z}^m$, where m is the number of weakly connected components in the graph.

We will be interested in a class of subsheaves of $\mathbb{Z}(G)$. Let H be a subgraph of G, we define $\mathbb{Z}(G)|_H$ to be the subsheaf of $\mathbb{Z}(G)$ such that:

- $F(v) = 0$ if, and only if, $v \notin V(H)$.
- $F(e) = 0$ if, and only if, $e \notin A(H)$.

In this case, the restriction map are either *id* or 0. The subsheaves encode the phenome of the subgraphs:

Proposition 9.6.13. *Let G be a weakly connected component. We have:*

$$\Gamma(G, \mathbb{Z}(G)|_H) = \begin{cases} \mathbb{Z} & \text{if } H = G \\ 0 & \text{otherwise} \end{cases}$$

Proof. If $H \neq G$, then some arc e in G is not present in H. Then every section s in $\Gamma(G, \mathbb{Z}(G)|_H)$ restricts to 0 on e. As all non-zero restriction maps are *id* and the graph is weakly connected, then s restricts to 0 on all other vertices and edges. Indeed, s restricts to 0 on $h(e)$ and $v(e)$. In turn, it restricts to 0 on all the arcs whose either endpoint is either $h(e)$ and $v(e)$. The *contagion* process goes on. By property *ii.* of the sheaf, we get that s is 0. □

Those subsheaves then enable us to define a lifting:

Proposition 9.6.14. *The functor L sending:*

- *A subgraph H to the morphism $\mathbb{Z}(G) \to \mathbb{Z}(G)/\mathbb{Z}(G)|_H$, and*

- *An inclusion $H \subseteq H'$ to a canonical diagram of sheaves:*

$$\begin{tikzcd}
\mathbb{Z}(G) \arrow[r, "id"] \arrow[d] & \mathbb{Z}(G) \arrow[d] \\
\mathbb{Z}(G)/\mathbb{Z}(G)|_H \arrow[r, "L(\subseteq)"] & \mathbb{Z}(G)/\mathbb{Z}(G)|_{H'}
\end{tikzcd}$$

is a lifting.

Proof. Condition (i.) is clearly met. To check condition (ii.), we have an commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}(G) \arrow[r] \arrow[d] & \mathbb{Z}(G) \oplus \mathbb{Z}(G) \arrow[r] \arrow[d] & \mathbb{Z}(G) \arrow[r] \arrow[d] & 0 \\
\mathbb{Z}(G) \arrow[r] & \mathbb{Z}(G)/\mathbb{Z}(G)|_H \oplus \mathbb{Z}(G)/\mathbb{Z}(G)|_{H'} \arrow[r] & \mathbb{Z}(G)/(\mathbb{Z}(G)|_H + \mathbb{Z}(G)|_{H'}) \arrow[r] & 0
\end{tikzcd}.$$

The top row is clearly exact. The bottom row can also be seen to be exact, through the use of Corollary 9.6.9 and Corollary 9.6.10. The sheaf $\mathbb{Z}(G)|_H + \mathbb{Z}(G)|_{H'}$ can be computed by taking linear sums of abelian groups (as subgroups of $\mathbb{Z}$) pointwise on the irreducible components. □

This lifting thus induces an abelian veil-lift:

Proposition 9.6.15. *The following commutative diagram:*

$$\begin{tikzcd}
\textbf{Ab-grp} & \textbf{Shv}(G)^{\mathbf{2}} \arrow[l, "\Gamma(G,-)\circ\ker"'] \\
2^{\{*\}} \arrow[u, "\text{free}"] & \textbf{Sub}(G) \arrow[l, "\Phi"'] \arrow[u, "L"']
\end{tikzcd}$$

is an abelian veil-lift.

Proof. The functor ker and $\Gamma(G, -)$ are both left exact. See [Ada17g] for a discussion on ker. All the remaining required pieces have been derived in this section. □

Furthermore, this abelian veil-lift is specifically such that it preserves the common subgraph:

Proposition 9.6.16. *For every subgraphs H and H', the sequence of sheaves:*

$$0 \to \mathbb{Z}(G)|_{H \cap H'} \to \mathbb{Z}(G)|_H \oplus \mathbb{Z}(G)|_{H'} \to \mathbb{Z}(G)|_{H \cup H'} \to 0$$

is exact.

Proof. Exactness can be easily checked on the irreducible components, i.e. the edges and the vertices. □

Applying the global section functor may cause a loss exactness on the right, indicating generativity. This loss can be remedied by computing cohomology objects.

Recovering exactness.

We compute the derived functor of the global section functor. Only the first cohomology objects (or derived functor) is non-trivial as the graph is a Noetherian topological space with dimension 2.

To compute the objects, let F be a sheaf on the digraph $G(V, A)$, and define:

$$d_h := \bigoplus_{e \in A} F(e) \to \bigoplus_{v \in V} F(v) \qquad d_t := \bigoplus_{e \in A} F(e) \to \bigoplus_{v \in V} F(v)$$

Where d_h (resp. d_t) takes $F(e)$ to $F(he)$ (resp. $F(te)$). We then have $\Gamma(G, F) = \ker(d_h - d_t)$ and $\mathbb{H}(F) := R^1\Gamma(G, F) = \operatorname{coker}(d_h - d_t)$. We refer the reader to [Fri11] Section 1.2.1 for a discuss on the cohomology groups of a sheaf defined over a graph, and Section 1.4.2 for an injective resolution of a sheaf F.

Given a short exact sequence of sheaves:

$$0 \to \mathbb{Z}(G)|_{H \cap H'} \to \mathbb{Z}(G)|_H \oplus \mathbb{Z}(G)|_{H'} \to \mathbb{Z}(G)|_{H \cup H'} \to 0$$

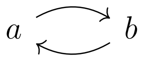
We get a six-term exact sequence in cohomology:

$$0 \to \Gamma(G, \mathbb{Z}(G)|_{H \cap H'}) \to \Gamma(G, \mathbb{Z}(G)|_H) \oplus \Gamma(G, \mathbb{Z}(G)|_{H'}) \to \Gamma(G, \mathbb{Z}(G)|_{H \cup H'}) \to \cdots$$
$$\cdots \to \mathbb{H}\big(\mathbb{Z}(G)|_{H \cap H'}\big) \to \mathbb{H}\big(\mathbb{Z}(G)|_H \oplus \mathbb{H}\big(\mathbb{Z}(G)|_{H'}\big) \to \mathbb{H}\big(\mathbb{Z}(G)|_{H \cup H'}\big) \to 0$$

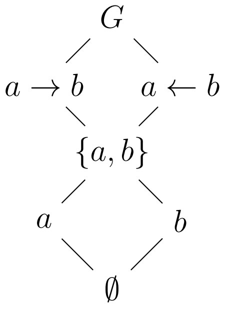
Of course, $\mathbb{H}$ is not the (first) derived functor of $\Gamma(G, -) \circ \ker$. The functor ker is also not exact.

Concrete example.

For instance, let G be a cycle graph as follows:



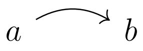
The lattice of open sets is depicted through a Hasse diagram. The *vertices* of the diagram refer to the open sets, and the edges refers to an inclusion of open sets.

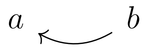


The lattice is distributive, and its join irreducible elements are a,b (corresponding to the two vertices) and $a \to b$, $a \leftarrow b$ (corresponding to the two arcs). The sheaf $\mathbb{Z}(G)$ can then be represented on the Hasse diagram as:

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbb{Z} \arrow[dr, no head] & & \mathbb{Z} \arrow[dl, no head] \\
 & \mathbb{Z} \oplus \mathbb{Z} \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbb{Z} \arrow[dr, no head] & & \mathbb{Z} \arrow[dl, no head] \\
 & 0 &
\end{tikzcd}$$

The global section is indeed $\mathbb{Z}$ as can be seen on the topmost element. Let H and H' be the following two subgraphs:





Their corresponding sheaves $\mathbb{Z}(G)|_H$ and $\mathbb{Z}(G)|_{H'}$ can be represented as:

$$\begin{tikzcd}
 & 0 \arrow[dl, no head] \arrow[dr, no head] & & & & 0 \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbb{Z} \arrow[dr, no head] & & 0 \arrow[dl, no head] & & 0 \arrow[dr, no head] & & \mathbb{Z} \arrow[dl, no head] \\
 & \mathbb{Z} \oplus \mathbb{Z} \arrow[dl, no head] \arrow[dr, no head] & & & & \mathbb{Z} \oplus \mathbb{Z} \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbb{Z} \arrow[dr, no head] & & \mathbb{Z} \arrow[dl, no head] & & \mathbb{Z} \arrow[dr, no head] & & \mathbb{Z} \arrow[dl, no head] \\
 & 0 & & & & 0 &
\end{tikzcd}$$

Their *common* sheaf $\mathbb{Z}(G)|_{H \cap H'}$ can be represented as:

$$\begin{tikzcd}
 & 0 \arrow[dl, no head] \arrow[dr, no head] & \\
0 \arrow[dr, no head] & & 0 \arrow[dl, no head] \\
 & \mathbb{Z} \oplus \mathbb{Z} \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbb{Z} \arrow[dr, no head] & & \mathbb{Z} \arrow[dl, no head] \\
 & 0 &
\end{tikzcd}$$

For every proper subsheaf of $\mathbb{Z}(G)$, the group of global section is the 0 group. We can then form the following exact sequence:

$$0 \to \mathbb{Z}(G)|_{H \cap H'} \to \mathbb{Z}(G)|_H \oplus \mathbb{Z}(G)|_{H'} \to \mathbb{Z}(G)|_{H \cup H'} \to 0$$

Computing the cohomology objects, and forming the long exact sequence yields:

$$0 \to 0 \to 0 \oplus 0 \to \Gamma(G, \mathbb{Z}(G)|_{H \cup H'}) \to \mathbb{Z}^2 \xrightarrow{(\mathbb{H}(i), \mathbb{H}(i'))} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{H}(\mathbb{Z}(G)|_{H \cup H'}) \to 0$$

As $\mathbb{H}(i)$ and $\mathbb{H}(i')$ have to be (by symmetry) isomorphic (non-zero) maps, we get $\Gamma(G, \mathbb{Z}(G)|_{H \cup H'}) = \mathbb{Z}$ and $\mathbb{H}(\mathbb{Z}(G)|_{H \cup H'}) = \mathbb{Z}$.

Detecting a subgraph. Detecting subgraphs may be achieved through the use of the (direct or inverse) image functors. Indeed, a subgraph of G is, by definition, only an open subset of the topological space underlying G. The details of this approach will not be further pursued in this chapter.

In general topological spaces. Directed graphs can be replaced by connected topological spaces. The problem then becomes that of deciding whether subspaces cover the whole space or not.

We can again define the constant sheaf $\mathbb{Z}(X)$ on the whole space X, and consider corresponding subsheaves to represent the subspaces of X along the lines of what was done in this section.

Another source of abelian veils. Rather than deciding whether coverings are full or not, we can consider a continuous map $f : X \to Y$. The map lifts to functor $f^* : \mathbf{Shv}(Y) \to \mathbf{Shv}(X)$. The functor admits a right adjoint $f_* : \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$. The functor f_* is a left-exact functor, and can be used in certain abelian lifts. The chapter will however not pursue this direction.

9.7 All veils admit an abelian veil lift.

The following section is derived as a consequence of the development on liftings and veil-lifts in [Ada17g]. We have defined a presheaf to be a functor from $\mathcal{O}$ to $\mathbf{Ab\text{-}grp}^{op}$. The definition of a presheaf extends to have an arbitrary lattice (and more generally a category) as a base space. Presheaves can also be dualized. We will use copresheaves to define an abelian veil lift for an arbitrary veil.

Let $\mathcal{L}$ be a lattice. A presheaf on $\mathcal{L}$ taking values in $\mathcal{D}$ is a contravariant functor $\mathcal{L} \to \mathcal{D}$. A copresheaf on $\mathcal{L}$ taking values in $\mathcal{D}$ is a covariant functor $\mathcal{L} \to \mathcal{D}$. The category of copresheaves on $\mathcal{L}$ taking values in $\mathcal{D}$ is denoted by $\mathcal{L}^{\mathcal{D}}$.

Proposition 9.7.1. *The functor $h : \mathcal{L} \to \mathcal{L}^{\mathbf{Set}}$ sending:*

- *An element s to $- \mapsto \hom(s, -)$*
- *An inclusion $s \leq s'$ to the unique morphism of copresheaves $h(s) \to h(s')$*

is a lifting (not abelian) in the sense of [Ada17g].

Proof. The functor h is known as the Yoneda embedding (see e.g. [ML98] Ch III) and is known the satisfy the required properties. □

We define free : $\mathcal{L}^{\mathbf{Set}} \to \mathcal{L}^{\mathbf{Ab\text{-}grp}}$ to be the functor such that for every copresheaf F, we have that $\operatorname{free} F(e) = \mathbb{Z}(F(e))$ is the free abelian group generated by $F(e)$.

Proposition 9.7.2. *The functor* $\operatorname{free} \circ h$ *is an abelian lifting.*

Proof. The functor h preserves connected colimits in the case of preorders, and free preserves all colimits being a left adjoint. Their composition $\operatorname{free} \circ h$ then satisfies property (ii) of an abelian lifting. Property (i) follows as both h and free reflect isomorphisms. Indeed, every isomorphism on free abelian groups is induced by an isomorphism (i.e., a bijection) on their generating sets. □

This lifting induces a veil-lift:

Theorem 9.7.3. *The following commutative diagram:*

$$\begin{tikzcd}
\mathbf{Ab\text{-}grp}^{\boldsymbol{P}} & & \mathbf{Ab\text{-}grp}^{\boldsymbol{S}} \arrow[ll, "\Phi_!"'] \\
\boldsymbol{P} \arrow[u, "\operatorname{free} \circ h"] & & \boldsymbol{S} \arrow[ll, "\Phi"'] \arrow[u, "\operatorname{free} \circ h"']
\end{tikzcd}$$

is an abelian veil-lift.

Proof. The details of the proof lie outside the scope of the chapter. We refer the reader to [Ada17g] for a proof of the statement. □

The construction of $\Phi_!$ lies outside the scope of the chapter, and refer the reader to [Ada17g] for the details. This lift, as it applies to every join-semilattice, is very general. The goal is to always find better lifts by exploiting the specific structure of the problem.

9.8 Concluding remarks.

Various other liftings may be found, and many situations provided in this chapter may be further generalized. Abelian lifts are discussed in more generality in [Ada17g]. The goal hereon is to develop adequate lifts for various situtations generative effects or classes of veils.

Chapter 10

Conclusion

We begin by ending the thesis with three remarks.

The notion, and the definition, of generative effects depends neither on the question asked nor on the answer to that question. As such, starting with the notion of generative effects, there are multiple questions that may be asked. Given however the fundamental presence of the inequality (or non-isomorphism), once the veil is applied, we view the most natural and fundamental question as that of mending it. We thus pursued this particular direction. Yet many other directions are also very much worth pursuing.

Homological methods (and thus the use of abelian veil lifts) are not the only means to relate the behavior of the system to its other constituents, namely to answer the particular question posed. There are other (extended) means, e.g., either through non-abelian homological algebra or homotopical methods.

Finally, the question and thus very likely the answer may admit a different mathematical interpretation. In this thesis, they rely on a loss of exactness and then on a fix of the loss. It may be possible to replace colimits and non-preservation of colimits by the veil by another operation and non-commutativity (or non-preservation) condition. The form of the question remains unchanged, but the mathematics may change. Also the *physical* or *intuitive* interpretation of the phenomenon may also change, and would need to be adapted accordingly.

10.1 Further directions.

The developed theory is solid, but it has much to evolve and grow into its full potential. We next describe three rough directions of future investigation.

10.1.1 More connections to systems, risk and failures.

The practical boundary of the theory should be further pushed. The theory already accounts for several classes of examples. Its reaches can be further expanded to more realistic models of cascading phenomena and its tools can be further refined to deal with large scale issues. The theory of D-modules, and algebraic analysis, already put to use in systems theory may provide us with a good route. It would also immensely benefit from settling important cases. The list of potential directions along those lines is non-exhaustive.

10.1.2 Further algebraic development.

The homological algebra development barely reaches the full development in the field. The use of derived categories should be come into play to elucidate the situation more. Grothendieck topologies, sheaves and toposes ought to play a more prominent role. Many ideas from algebraic geometry and scheme theory can be incorporated. Spectral sequences can be further brought into the picture to enhance the computational power of the theory. Homological methods are not the only means to recover the link. Non-abelian methods and simplicial methods also extend the scope. Homotopical algebra then gains a place. Elements from K-theory will also prove to be very relevant. The language in the thesis has intentionally relied on only most elementary concepts in functorial language. Much is to be gained from more intricate constructs, not to mention higher category theory. The goal of such a thrust should not be seen as a mathematical exercise. The development needs to be well attached to a systems-theoretic intuition and interpretation.

10.1.3 Syntax, semantics and languages.

There is much to understand in the separation and interplay between syntax and semantics in systems theoretic issues. J. C. Willems' behavioral approach to systems theory is advocating constructions of systems in a topos (thus cartesian closed) via colimits. The category of systems (in the behavioral sense) then immediately admits an internal language, coinciding with intuitionistic logic. This realization provides a glimpse into a formal language of systems and a formal analogy between physical systems and programs. The interplay of languages, syntax and semantics has much to offer in providing a sound theory of modeling. Many ideas of model theory and categorical logic ought to come into picture.

The languages arising in this context are very fitting to a theory of interaction. Our intent goes further into giving a vivid understanding and development of languages suited for the purpose of understanding interaction-related events. The goal in this respect consists of figuring out ways (if ever possible) to capture syntactically homological ideas or intuition.

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